Math 2550-01

2/22/24

Ex continued...

Last time we showed that the solutions to

$$\boxed{\begin{aligned} 5x_1 - 2x_2 + 6x_3 &= 0 \\ -2x_1 + x_2 + 3x_3 &= 1 \end{aligned}} \quad (*)$$

are given by

$$\boxed{\begin{aligned} x_1 &= 2 - 12t \\ x_2 &= 5 - 27t \\ x_3 &= t \end{aligned} \qquad \text{where } t \text{ can be any real number}}$$

This means there are an infinite # of solutions to (*), one for each t.

t	$x_1 = 2-12t$	$x_2 = 5-27t$	$x_3 = t$
1	-10	-22	1
0	2	5	0
$1/3$	-2	-4	$1/3$
π	$2-12\pi$	$5-27\pi$	π
⋮	⋮	⋮	⋮

$x_1 = -10$
$x_2 = -22$
$x_3 = 1$
is a sol. to (*)

another sol. to (*) is
$x_1 = 2$
$x_2 = 5$
$x_3 = 0$

Ex: Solve

$$\boxed{\begin{aligned} a + 3b - 2c \quad\quad + 2e \quad\quad &= 0 \\ 2a + 6b - 5c - 2d + 4e - 3f &= -1 \\ 5c + 10d \quad\quad + 15f &= 5 \\ 2a + 6b \quad\quad + 8d + 4e + 18f &= 6 \end{aligned}}$$

already has a 1 here

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right)$$

make these 0

$-2R_1 + R_2 \to R_2$

$-2R_1 + R_4 \to R_4$

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

make 1

$$-R_2 \to R_2$$

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

make these 0

$$-5R_2 + R_3 \to R_3$$

$$-4R_2 + R_4 \to R_4$$

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right)$$

$R_3 \leftrightarrow R_4$

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$1/6\, R_3 \to R_3$

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

in row echelon form

Back to the systems:

$$\begin{aligned}
a + 3b - 2c \qquad\quad + 2e \qquad &= 0 \quad (1)\\
c + 2d \qquad + 3f &= 1 \quad (2)\\
f &= 1/3 \quad (3)\\
0 &= 0
\end{aligned}$$

leading variables: a, c, f

free variables: b, d, e

Solve for leading variables and give free variables new names:

$$\begin{aligned}
a &= -3b + 2c - 2e & (1)\\
c &= 1 - 2d - 3f & (2)\\
f &= 1/3 & (3)\\
b &= s & (4)\\
d &= t & (5)\\
e &= u & (6)
\end{aligned}$$

Back-substitute:

(6) $e = u$

(5) $d = t$

(4) $b = s$

(3) $f = 1/3$

(2) $c = 1 - 2d - 3f = 1 - 2t - 3\left(\frac{1}{3}\right)$
$= -2t$

(1) $a = -3b + 2c - 2e$
$= -3s + 2(-2t) - 2u$
$= -3s - 4t - 2u$

$\boxed{\text{Answer:}}$

$a = -3s - 4t - 2u$
$b = s$
$c = -2t$
$d = t$
$e = u$
$f = 1/3$

where s, t, u can be any real numbers

For example, if you set

$s = 4, t = 0, u = 1$

then we get

$a = -3(4) - 4(0) - 2(1) = -14$
$b = 4$
$c = -2(0) = 0$
$d = 0$
$e = 1$
$f = 1/3$

So, one solution to the system is
$a = -14, b = 4, c = 0, d = 0, e = 1, f = 1/3$

There are infinitely many more.

Theorem: A system of linear equations has either

(i) no solutions,

(ii) exactly one solution,

or (iii) infinitely many solutions.

Topic 4 – The inverse of a matrix

<u>Motivation:</u> For numbers, we have multiplicative inverses.

For example, $\underset{\uparrow}{5} \cdot \underset{\uparrow}{\frac{1}{5}} = 1.$

multiplicative inverses

Here $5^{-1} = \frac{1}{5}$.

Can we do the same thing for matrices?

<u>Recall:</u> $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

and so on...

Def: Let A be an $n \times n$ matrix. [So, A is a square matrix]

We say that A is <u>invertible</u> if there exists an $n \times n$ matrix B where

$$AB = I_n = BA$$

So,
$AB = I_n$
$BA = I_n$

If $AB = BA = I_n$, then we say that A and B are <u>inverses</u> of each other.

Ex: Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

and $B = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$.

Let's check if A and B are inverses of each other.

We have

$$AB = \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}}_{2 \times 2} \underbrace{\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}}_{2 \times 2}$$

$\checkmark$

answer is 2×2

$$= \begin{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} & \begin{pmatrix} 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} (1)(-1)+(1)(2) & (1)(1)+(1)(-1) \\ (2)(-1)+(1)(2) & (2)(1)+(1)(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Also,

$$BA = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (-1 \ 1) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} & (-1 \ 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ (2 \ -1) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} & (2 \ -1) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -1+2 & -1+1 \\ 2-2 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Since $AB = I_2$ and $BA = I_2$

We know that A and B are inverses of each other.