# Ex: Solve

$$\boxed{\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}}$$

we want a 1 here

make these into zeros

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

$$\begin{array}{rl} (-2 \quad -2 \quad -4 \mid -18) & \leftarrow -2R_1 \\ +\,(2 \quad 4 \quad -3 \mid 1) & \leftarrow R_2 \\ \hline (0 \quad 2 \quad -7 \mid -17) & \leftarrow \text{new } R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

$$\xrightarrow{-3R_1+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right)$$

put a 1 here

$$\begin{array}{rl} \left(\begin{array}{ccc|c} -3 & -3 & -6 & -27 \end{array}\right) & \leftarrow -3R_1 \\ +\left(\begin{array}{ccc|c} 3 & 6 & -5 & 0 \end{array}\right) & \leftarrow R_3 \\ \hline \left(\begin{array}{ccc|c} 0 & 3 & -11 & -27 \end{array}\right) & \leftarrow \text{new } R_3 \end{array}$$

$$\xrightarrow{\frac{1}{2}R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array}\right)$$

make this zero

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array}\right)$$

turn into 1

$$\xrightarrow{-3R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right)$$

$$\begin{array}{rl} & \left(\begin{array}{ccc|c} 0 & -3 & \frac{21}{2} & \frac{51}{2} \end{array}\right) \leftarrow -3R_2 \\ + & \left(\begin{array}{ccc|c} 0 & 3 & -11 & -27 \end{array}\right) \leftarrow R_3 \\ \hline & \left(\begin{array}{ccc|c} 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right) \leftarrow \text{new } R_3 \end{array}$$

$$\xrightarrow{-2R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right)$$

left side is in row echelon form

Now turn the reduced matrix back into equations.

$$\boxed{\begin{aligned} x + y + 2z &= 9 \quad (1)\\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \quad (2)\\ z &= 3 \quad (3)\end{aligned}}$$

leading variables are $x, y, z$

There are no free variables

Solve in terms of leading variables.

$$\boxed{\begin{aligned} x &= 9 - y - 2z \quad (1)\\ y &= -\tfrac{17}{2} + \tfrac{7}{2}z \quad (2)\\ z &= 3 \quad (3)\end{aligned}}$$

Now we back-substitute starting at the last equation and going upwards.

③ gives $\boxed{z=3}$

sub in $z=3$

② gives

$$y = -\frac{17}{2} + \frac{7}{2}z = -\frac{17}{2} + \frac{7}{2}(3)$$
$$= 2$$

So, $\boxed{y=2}$

sub in $z=3$ and $y=2$

① gives

$$x = 9 - y - 2z$$
$$= 9 - 2 - 2(3) = 1$$

So $\boxed{x=1}$

Thus, the only solution to the system is

$$\boxed{\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}} \quad \text{or} \quad \boxed{(x,y,z) = (1,2,3)}$$

Let's check the answer to make sure it works 35

Original System

$$\left.\begin{aligned} x+y+2z &= 9 \\ 2x+4y-3z &= 1 \\ 3x+6y-5z &= 0 \end{aligned}\right\} \begin{aligned} 1+2+2(3) &= 9 \checkmark \\ 2(1)+4(2)-3(3) &= 1 \checkmark \\ 3(1)+6(2)-5(3) &= 0 \checkmark \end{aligned}$$

$x=1,\ y=2,\ z=3$ works

This is the only solution to the system. There is no other solution (*)

Ex: Solve

$$\begin{aligned} -2b+3c&=1\\ 3a+6b-3c&=-2\\ 6a+6b+3c&=5 \end{aligned}$$

want a 1 here

$$\left(\begin{array}{ccc|c} 0 & -2 & 3 & 1\\ 3 & 6 & -3 & -2\\ 6 & 6 & 3 & 5 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 3 & 6 & -3 & -2\\ 0 & -2 & 3 & 1\\ 6 & 6 & 3 & 5 \end{array}\right)$$

put ~~zeros~~ zeros here

$$\xrightarrow{\frac{1}{3}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3\\ \boxed{0} & -2 & 3 & 1\\ \boxed{6} & 6 & 3 & 5 \end{array}\right)$$

$$\xrightarrow{-6R_1+R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3\\ 0 & -2 & 3 & 1\\ 0 & -6 & 9 & 9 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & \boxed{-2} & 3 & 1 \\ 0 & -6 & 9 & 9 \end{array}\right)$$

put a 1 here

$$\xrightarrow{-\frac{1}{2}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & \boxed{-6} & 9 & 9 \end{array}\right)$$

make this a zero

$$\xrightarrow{6R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

this left side is in row echelon form

Now we turn it back into equations.

We get

$$\boxed{\begin{aligned} a + 2b - c &= -2/3 \\ b - 3/2\, c &= -1/2 \\ 0 &= 6 \end{aligned}}$$

Since we have $0 = 6$ in the last equation this tells us that the original system is <u>inconsistent</u> that is there are <u>no solutions</u> to the system.

Ex: Solve

$$\boxed{\begin{aligned} 5x_1 - 2x_2 + 6x_3 &= 0 \\ -2x_1 + x_2 + 3x_3 &= 1 \end{aligned}}$$

put a 1 here

$$\left(\begin{array}{ccc|c} \boxed{5} & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{array}\right)$$

$$\xrightarrow{2R_2 + R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 12 & 2 \\ \boxed{-2} & 1 & 3 & 1 \end{array}\right)$$

Could have instead done $\frac{1}{5}R_1 \to R_1$

make this a zero

$$\xrightarrow{2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{array}\right)$$

this left side is in row echelon form

Turn it back into equations.

$$\boxed{\begin{aligned} x_1 \quad + 12x_3 &= 2 \quad (1) \\ x_2 + 27x_3 &= 5 \quad (2) \end{aligned}}$$

leading variables are $x_1, x_2$.

free variable is $x_3$

Solve in terms of leading variables.

$$\boxed{\begin{aligned} x_1 &= 2 - 12x_3 \quad (1) \\ x_2 &= 5 - 27x_3 \quad (2) \end{aligned}}$$

Give the free variables a new name.

Let $\boxed{x_3 = t}$

Now backsubstitute.

② gives $x_2 = 5 - 27x_3$

$x_3 = t$

$$\boxed{x_2 = 5 - 27t}$$

① gives $x_1 = 2 - 12x_3$

$$\boxed{x_1 = 2 - 12t}$$

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**Answer**

$x_1 = 2 - 12t$
$x_2 = 5 - 27t$
$x_3 = t$

where $t$ can be any real number

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Infinitely many solutions, for example

$t = 1$

$x_1 = 2 - 12 = -10$
$x_2 = 5 - 27 = -22$
$x_3 = 1$

$t = 0$

$x_1 = 2 - 0 = 2$
$x_2 = 5 - 0 = 5$
$x_3 = 0$