

Ex: Solve

$$\begin{aligned}
 x + y + 2z &= 9 \\
 2x + 4y - 3z &= 1 \\
 3x + 6y - 5z &= 0
 \end{aligned}$$

✓
we want a 1 here

make these into zeros

$$\left(\begin{array}{ccc|c}
 1 & 2 & 2 & 9 \\
 2 & 4 & -3 & 1 \\
 3 & 6 & -5 & 0
 \end{array} \right)$$

$-2R_1 + R_2 \rightarrow R_2$

$$\left(\begin{array}{ccc|c}
 1 & 2 & 2 & 9 \\
 0 & 2 & -7 & -17 \\
 3 & 6 & -5 & 0
 \end{array} \right)$$

$$\begin{aligned}
 &(-2 \quad -2 \quad -4 \quad | \quad -18) \leftarrow -2R_1 \\
 + &(2 \quad 4 \quad -3 \quad | \quad 1) \leftarrow R_2 \\
 \hline
 &(0 \quad 2 \quad -7 \quad | \quad -17) \leftarrow \text{new } R_2
 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array} \right)$$

put a 1 here

$-3R_1 + R_3 \rightarrow R_3$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array} \right)$$



$$\begin{array}{l} (-3 \quad -3 \quad -6 \mid -27) \leftarrow -3R_1 \\ + (3 \quad 6 \quad -5 \mid 0) \leftarrow R_3 \\ \hline (0 \quad 3 \quad -11 \mid -27) \leftarrow \text{new } R_3 \end{array}$$

$\frac{1}{2}R_2 \rightarrow R_2$

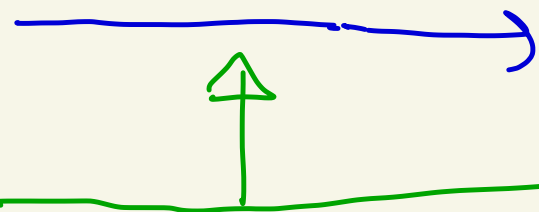
$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array} \right)$$

make this zero

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array} \right)$$

turn into 1

$-3R_2 + R_3 \rightarrow R_3$



$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array} \right)$$

$(0 \ -3 \ \frac{21}{2} \ | \ \frac{51}{2}) \leftarrow -3R_2$
 $+ (0 \ 3 \ -11 \ | \ -27) \leftarrow R_3$

 $(0 \ 0 \ -\frac{1}{2} \ | \ -\frac{3}{2}) \leftarrow \text{new } R_3$

$-2R_3 \rightarrow R_3$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array} \right)$$

left side is in row echelon form

Now turn the reduced matrix back into equations.

(33)

$$\begin{array}{l} x + y + 2z = 9 \quad (1) \\ y - \frac{7}{2}z = -\frac{17}{2} \quad (2) \\ z = 3 \quad (3) \end{array}$$

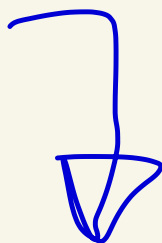
Leading variables are x, y, z

There are no free variables

Solve in terms of leading variables.

$$\begin{array}{l} x = 9 - y - 2z \quad (1) \\ y = -\frac{17}{2} + \frac{7}{2}z \quad (2) \\ z = 3 \quad (3) \end{array}$$

Now we back-substitute starting at the last equation and going upwards.



③ gives $z = 3$

② gives $y = -\frac{17}{2} + \frac{7}{2}z = -\frac{17}{2} + \frac{7}{2}(3) = 2$

sub in $z=3$

So, $y = 2$

sub in $z=3$ and $y=2$

① gives $x = 9 - y - 2z = 9 - 2 - 2(3) = 1$

So $x = 1$

Thus, the only solution to the system is

$x = 1$
 $y = 2$
 $z = 3$

or $(x, y, z) = (1, 2, 3)$

Let's check the answer to make sure it works

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Original System

$$\left. \begin{array}{l} x + y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 6y - 5z = 0 \end{array} \right\} \begin{array}{l} 1 + 2 + 2(3) = 9 \checkmark \\ 2(1) + 4(2) - 3(3) = 1 \checkmark \\ 3(1) + 6(2) - 5(3) = 0 \checkmark \end{array}$$

$x = 1, y = 2, z = 3$ works

This is the only solution to the system.

There is no other solution

(*)

Ex: Solve

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$$-2b + 3c = 1$$

$$3a + 6b - 3c = -2$$

$$6a + 6b + 3c = 5$$

want
a 1
here

$$\begin{pmatrix} 0 & -2 & 3 & | & 1 \\ 3 & 6 & -3 & | & -2 \\ 6 & 6 & 3 & | & 5 \end{pmatrix}$$

$$R_1 \leftrightarrow R_2 \rightarrow \begin{pmatrix} 3 & 6 & -3 & | & -2 \\ 0 & -2 & 3 & | & 1 \\ 6 & 6 & 3 & | & 5 \end{pmatrix}$$

put
zeros
here

$$\frac{1}{3}R_1 \rightarrow R_1 \rightarrow \begin{pmatrix} 1 & 2 & -1 & | & -2/3 \\ 0 & -2 & 3 & | & 1 \\ 6 & 6 & 3 & | & 5 \end{pmatrix}$$

$$-6R_1 + R_3 \rightarrow R_3 \rightarrow \begin{pmatrix} 1 & 2 & -1 & | & -2/3 \\ 0 & -2 & 3 & | & 1 \\ 0 & -6 & 9 & | & 9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{array} \right)$$

put a 1 here

$-\frac{1}{2}R_2 \rightarrow R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & -6 & 9 & 9 \end{array} \right)$$

make this a zero

$6R_2 + R_3 \rightarrow R_3$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & 0 & 6 \end{array} \right)$$

this left side is in row echelon form

Now we turn it back into equations.

We get

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$$a + 2b - c = -2/3$$

$$b - 3/2c = -1/2$$

$$0 = 6$$

→ Since we have $0 = 6$ in the last equation this tells us that the original system is inconsistent that is there are no solutions to the system.

Ex: Solve

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$$\begin{cases} 5x_1 - 2x_2 + 6x_3 = 0 \\ -2x_1 + x_2 + 3x_3 = 1 \end{cases}$$

put a 1 here

$$\left(\begin{array}{ccc|c} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{array} \right)$$

$2R_2 + R_1 \rightarrow R_1$

$$\left(\begin{array}{ccc|c} 1 & 0 & 12 & 2 \\ -2 & 1 & 3 & 1 \end{array} \right)$$

make this a zero

could have instead done $\frac{1}{5}R_1 \rightarrow R_1$

$2R_1 + R_2 \rightarrow R_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{array} \right)$$

this left side is in row echelon form

Turn it back into equations.

(40)

$$\begin{array}{l} x_1 + 12x_3 = 2 \quad (1) \\ x_2 + 27x_3 = 5 \quad (2) \end{array}$$

leading variables are x_1, x_2 .

free variable is x_3

Solve in terms of leading variables.

$$\begin{array}{l} x_1 = 2 - 12x_3 \quad (1) \\ x_2 = 5 - 27x_3 \quad (2) \end{array}$$

Give the free variables a new name.

Let $x_3 = t$

Now backsubstitute.

② gives $x_2 = 5 - 27x_3$

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$$x_2 = 5 - 27t$$

$$x_3 = t$$

① gives $x_1 = 2 - 12x_3$

$$x_1 = 2 - 12t$$

Answer

$$x_1 = 2 - 12t$$

$$x_2 = 5 - 27t$$

$$x_3 = t$$

where

t can

be any

real number

In finitely many solutions, for example

$$t = 1$$

$$x_1 = 2 - 12 = -10$$

$$x_2 = 5 - 27 = -22$$

$$x_3 = 1$$

$$t = 0$$

$$x_1 = 2 - 0 = 2$$

$$x_2 = 5 - 0 = 5$$

$$x_3 = 0$$