Math 2550-01
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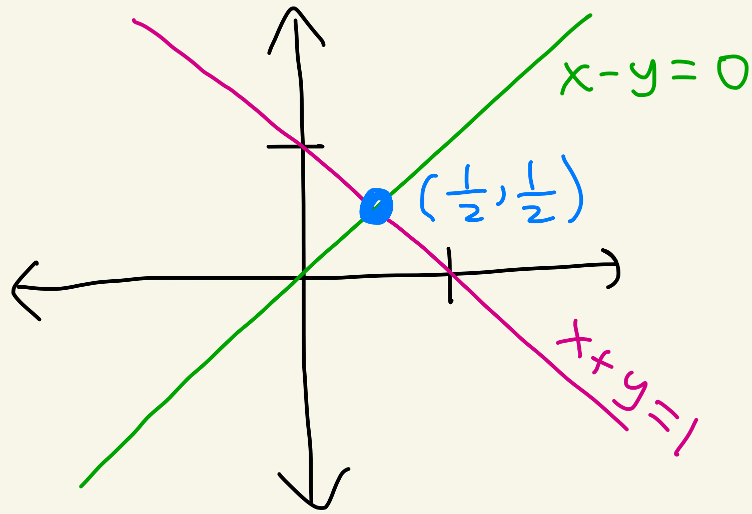
Theorem: Applying an elementary row operation to a system of linear equations does not change the solution space of the system.

Ex:

system

$$x+y=1$$
$$x-y=0$$



solution space is $(x,y)=\left(\frac{1}{2},\frac{1}{2}\right)$

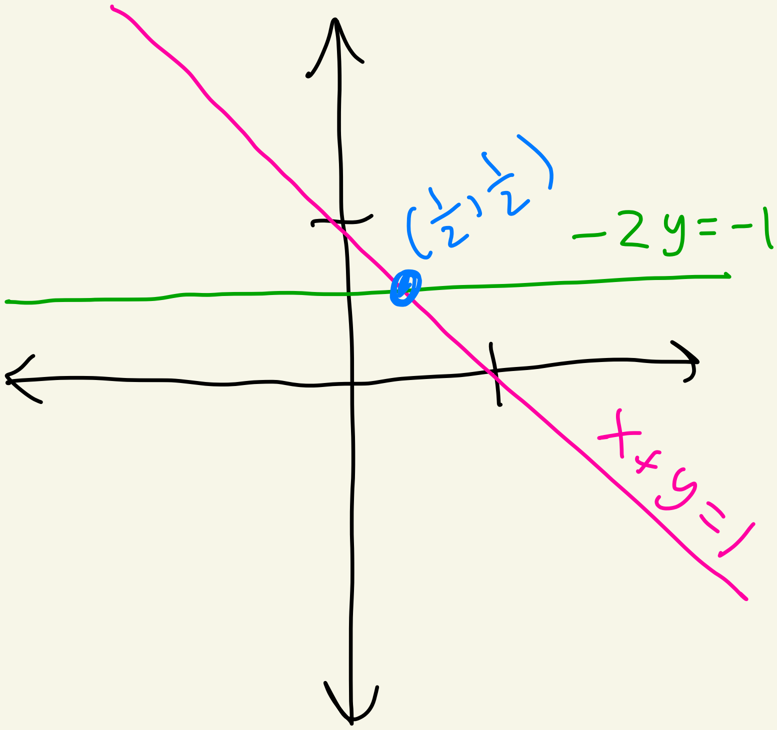
Let's apply an elementary row operation to this system and see what happens.

$$\underbrace{\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 0 \end{array}\right)}_{\text{matrix for } \begin{array}{l} x+y=1 \\ x-y=0 \end{array}} \xrightarrow{-R_1+R_2 \to R_2} \underbrace{\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & -1 \end{array}\right)}_{\text{new system}}$$

$$\begin{array}{rl} (-1 \;\; -1 \;|\; -1) & \leftarrow \boxed{-R_1} \\ +\,(1 \;\; -1 \;|\; 0) & \leftarrow \boxed{R_2} \\ \hline (0 \;\; -2 \;|\; -1) & \leftarrow \boxed{\text{new } R_2} \end{array}$$

new system

$$\boxed{\begin{array}{r} x+y=1 \\ -2y=-1 \end{array}}$$



The systems changed but not the solution of $(x,y)=(\frac{1}{2},\frac{1}{2})$

Idea of Gaussian elimination will be to ~~keep~~ applying elementary row operations until we can solve the system.
The solutions will not change as we do this.
The question is how to do this.
Thats what we will learn next.

Def: If a row of a matrix does not consist entirely of zeros then the leading entry in that row is the first non-zero entry when scanning from left to right.

Ex:

$$\begin{array}{cl} \begin{pmatrix} 2 & 1 & 0 & -1 \\ 0 & 0 & 3 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \leftarrow \text{row } 3 \\ \leftarrow \text{row } 4 \end{array} \end{array}$$

leading entry in row 1 is 2
leading entry in row 2 is 3
leading entry in row 3 is 5
row 4 doesn't have a leading entry.

Def: A matrix is in row echelon form if the following are true:

① If there are any rows that consist entirely of zeros, then those rows are grouped together at the bottom of the matrix.

② In any two consecutive rows that do not consist entirely of zeros, then the leading entry in the lower row occurs farther to the right than the leading entry in the upper row.

and

③ If a row does not consist entirely of zeros, then the leading entry of that row is 1.

A matrix is in reduced row echelon form if ①, ②, and ③ above are true and the following is true:

④ Each column that contains a leading 1 has zeros everywhere else in that column.

Ex:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

leading entries are circled

① ✓

② ✓

③ ✗

not in row echelon form

Ex:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

① ✓

② ✓

③ ✓

matrix is in row echelon form

④ ✗

matrix is not in reduced row echelon form

Ex:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

① ✓
② ✓
③ ✓
④ ✓

matrix is in reduced row echelon form

Ex:

$$\begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

① ✓
② X
③ ✓

not in row echelon form

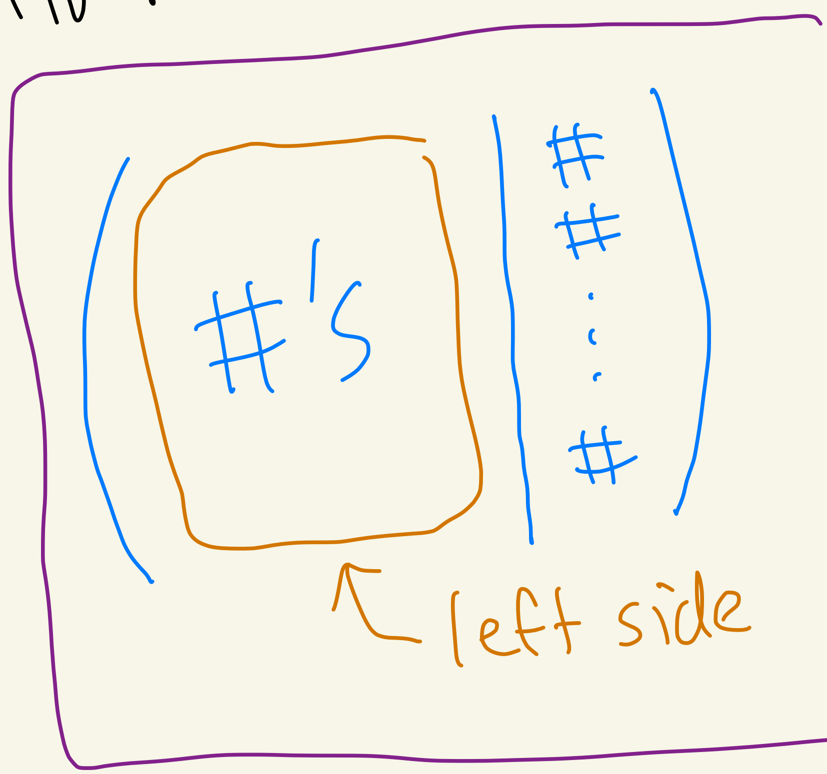
Ex:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

① ✓
② ✓
③ ✓

row echelon form ✓

Def: Suppose you have an augmented matrix for a system of linear equations.
Suppose you use elementary row operations to put the left side of the matrix into row echelon form.



The variable corresponding to the leading entry of a row is called a **leading variable** (or a **pivot variable**).
Any variable that doesn't occur as a leading variable is called a **free variable**.

Ex:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

left side is in row echelon form

leading entries are circled

Suppose the above matrix corresponds to this system:

$$\begin{aligned} x + z &= 3 \\ y + z &= 2 \\ 0 &= 0 \end{aligned}$$

leading variables are x, y

free variable is z

Method of Gaussian Elimination

① Put your system into an augmented matrix.

② Use elementary row operations to put the left side of the augmented matrix into row echelon form.

③ Write down the new system corresponding to the matrix.

④

<u>case (i):</u> If one of the equations is $0 = c$ where $c \neq 0$, then the system has no solutions.

<u>case (ii):</u> If case i doesnt

happen, then we can use back-substitution to find all the solutions, as follows:

(a) Solve the equations for the leading variables.

(b) Assign each free variable a new name as it can take on any value.

(c) Beginning with the last equation and working upwards, successively substitute each equation into the equation above it.