Math 2550

2/13/24

## Topic 3 – Systems of linear equations

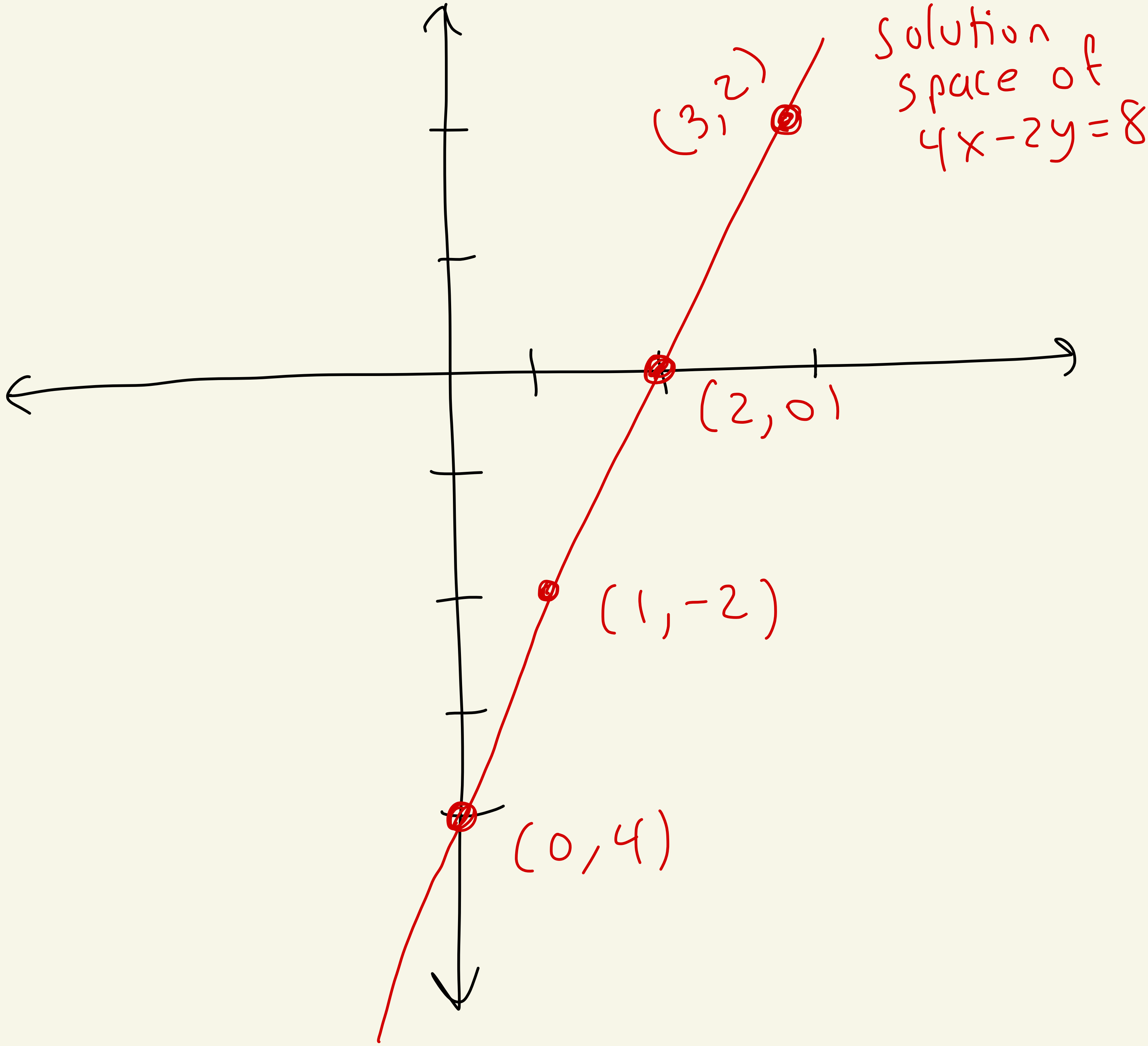
Def: A linear equation in the $n$ variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b} \quad (*)$$

where $a_1, a_2, \ldots, a_n, b$ are constant real numbers.

The solution space of $(*)$ consists of the set of all $(x_1, x_2, \ldots, x_n)$ that solve $(*)$.

**Ex:** $\underbrace{4x - 2y = 8}_{a_1 x_1 + a_2 x_2 = b}$ $\leftarrow$ lin. eq. in 2 variables $x, y$



Ex: Some more linear eqns:

$$5x_1 - 10x_2 + \frac{1}{2}x_3 = 7$$

$$\sqrt{2}\,x + y - 3w + \frac{1}{3}z = 0$$

Ex: Some non-linear eqns:

$$x^2 + y = 6$$

$$5\cos(x) + y = 4$$

Def: A system of m linear equations in n unknowns $x_1, x_2, \dots, x_n$ is a list of m equations of the form:

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}} \quad (*)$$

where the $a_{ij}$ are constant real #s.

The augmented matrix for $(*)$ is

$$\begin{array}{c} \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 \text{ column} & x_2 \text{ column} & x_n \text{ column} & \text{represents the = sign} \end{array} \end{array}$$

The <u>solution space</u> of the system (*) consists of all $(x_1, x_2, \ldots, x_n)$ that simultaneously solve all m equations. That is, the common solutions to all m equations.

Ex:

$$x + 2y = 3$$
$$4x + 5y = 6$$

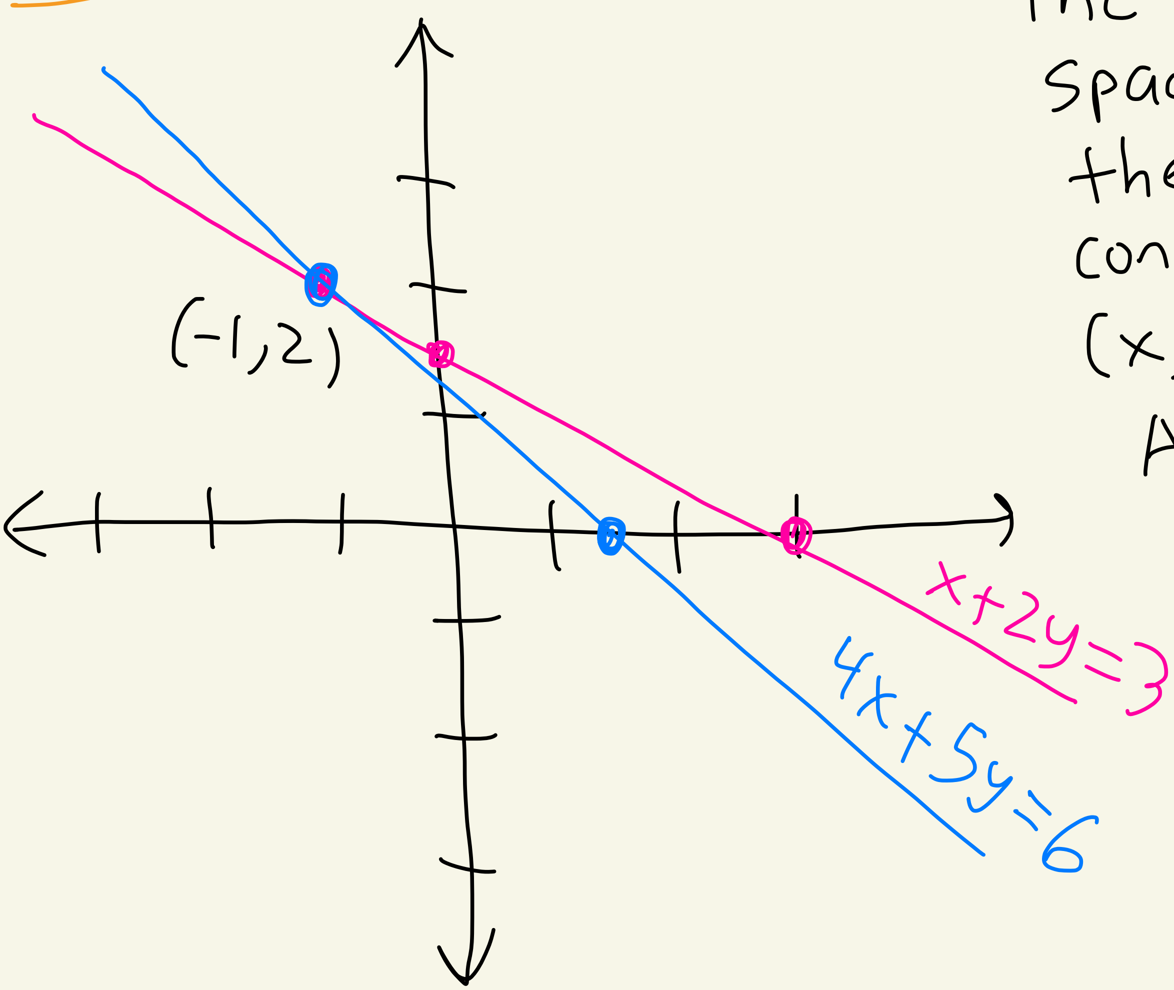
system of linear equations

$m = 2$ linear equations

$n = 2$ unknowns

Augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right)$$



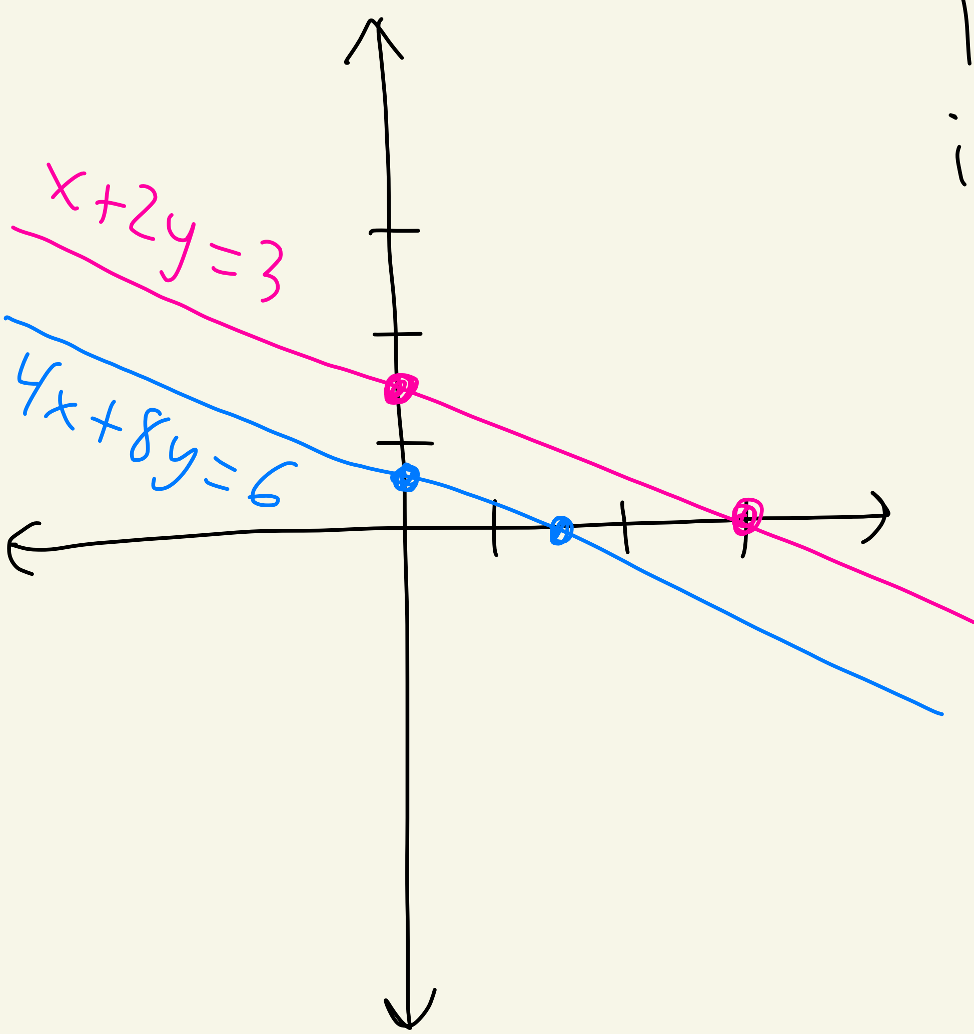

The solution space for the system consists of $(x,y) = (-1,2)$

As a set it is $\{(-1,2)\}$

Ex: Consider the system

$$\left.\begin{aligned} x + 2y &= 3 \\ 4x + 8y &= 6 \end{aligned}\right]$$ system of lin. eqns. $m=2$, $n=2$ unknowns

Augmented matrix: $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 8 & 6 \end{array}\right)$



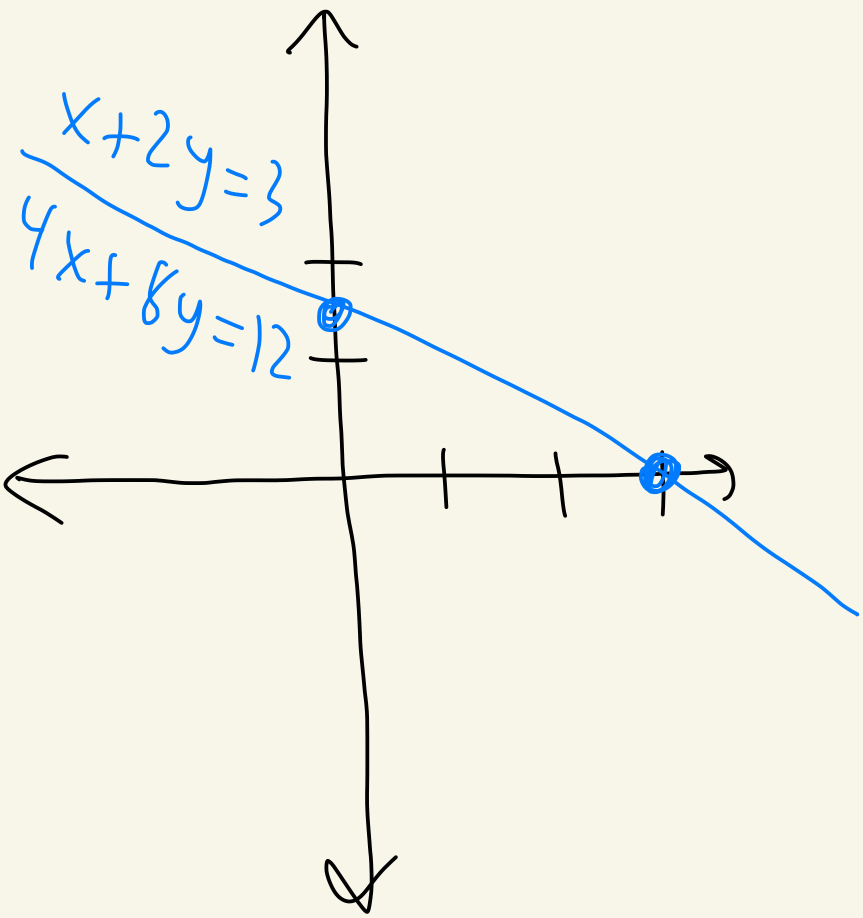

These lines don't intersect. So the solution space is empty! There are no solutions to the system.

Ex:

$$x + 2y = 3$$
$$4x + 8y = 12$$

$m = 2$ lin. eqns.
$n = 2$ unknowns.

Augmented matrix: $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 8 & 12 \end{array}\right)$




Its the same line twice. The solution space is all $(x, y)$ that solve $x + 2y = 3$. It's an infinite solution space.

Ex:

$$\boxed{\begin{aligned} x + y + 2z &= 9 \\ 2x \qquad - 3z &= 1 \\ -x + 6y - 5z &= 0 \end{aligned}}$$

$m = 3$ lin. eqns
$n = 3$ unknowns

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 0 & -3 & 1 \\ -1 & 6 & -5 & 0 \end{array}\right)$$

$\boxed{\text{Ex:}}$

$$\boxed{\begin{aligned} 2y - 3w &= 5 \\ 4x + y + w - 2z &= 0 \\ x - y + w &= 1 \end{aligned}}$$

3 eqns
4 unknowns

Augmented matrix

$$\left(\begin{array}{cccc|c} 0 & 2 & -3 & 0 & 5 \\ 4 & 1 & 1 & -2 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{array}\right)$$

Def: Given a system of linear equations there are three operations that we call elementary row operations

They are:

1. Multiply one of the rows/equations by a non-zero constant
2. Interchange two rows/equations
3. Add a multiple of one row/equation to another row/equation

Ex: (Multiply row/equation by non-zero constant)

$\boxed{\text{Equation viewpoint}}$

$$\boxed{\begin{aligned} 3x - 6y + 9z &= 2 \\ x \phantom{- 6y} + z &= 5 \\ 2x - y - z &= -1 \end{aligned}} \xrightarrow{\frac{1}{3}R_1 \to R_1} \boxed{\begin{aligned} x - 2y + 3z &= \tfrac{2}{3} \\ x \phantom{- 2y} + z &= 5 \\ 2x - y - z &= -1 \end{aligned}}$$

$\boxed{\text{Matrix viewpoint}}$

$$\left(\begin{array}{ccc|c} 3 & -6 & 9 & 2 \\ 1 & 0 & 1 & 5 \\ 2 & -1 & -1 & -1 \end{array}\right) \xrightarrow{\frac{1}{3}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2/3 \\ 1 & 0 & 1 & 5 \\ 2 & -1 & -1 & -1 \end{array}\right)$$

Ex: (Interchange two rows/equations)

$\boxed{\text{Equation viewpoint}}$

$$\boxed{\begin{aligned} 3x - 6y + 9z &= 2 \\ x \phantom{- 6y} + z &= 5 \\ 2x - y - z &= -1 \end{aligned}} \xrightarrow{R_1 \leftrightarrow R_2} \boxed{\begin{aligned} x \phantom{- 6y} + z &= 5 \\ 3x - 6y + 9z &= 2 \\ 2x - y - z &= -1 \end{aligned}}$$

$\boxed{\text{matrix viewpoint}}$

$$\left(\begin{array}{ccc|c} 3 & -6 & 9 & 2 \\ 1 & 0 & 1 & 5 \\ 2 & -1 & -1 & -1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 3 & -6 & 9 & 2 \\ 2 & -1 & -1 & -1 \end{array}\right)$$

Ex: (Add a multiple of one row to another row)

$\boxed{\text{equation viewpoint}}$

$$\boxed{\begin{aligned} x + y - z &= 1 \\ 2x + y + 2z &= 0 \\ x - y - z &= 3 \end{aligned}} \xrightarrow{-2R_1 + R_2 \to R_2} \boxed{\begin{aligned} x + y - z &= 1 \\ -y + 4z &= -2 \\ x - y - z &= 3 \end{aligned}}$$

$$\boxed{\begin{aligned} -2x - 2y + 2z &= -2 \\ + \quad 2x + y + 2z &= 0 \\ \hline 0x - y + 4z &= -2 \end{aligned}} \begin{aligned} &\leftarrow \boxed{-2R_1} \\ &\leftarrow \boxed{R_2} \\ &\leftarrow \boxed{\text{new } R_2} \end{aligned}$$

$\boxed{\text{matrix viewpoint}}$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 1 & 2 & 0 \\ 1 & -1 & -1 & 3 \end{array}\right) \xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -1 & 4 & -2 \\ 1 & -1 & -1 & 3 \end{array}\right)$$

$$\begin{array}{cl} & \left(\begin{array}{ccc|c} -2 & -2 & 2 & -2 \end{array}\right) \leftarrow \boxed{-2R_1} \\ + & \left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \end{array}\right) \leftarrow \boxed{R_2} \\ \hline & \left(\begin{array}{ccc|c} 0 & -1 & 4 & -2 \end{array}\right) \leftarrow \boxed{\text{new } R_2} \end{array}$$