Math 2550-01

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Def: Let $\vec{v}, \vec{w}$ be in $\mathbb{R}^n$.

Suppose

$$\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$$

$$\vec{w} = \langle b_1, b_2, \ldots, b_n \rangle$$

Define the dot product of $\vec{v}$ and $\vec{w}$ to be

$$\vec{v} \cdot \vec{w} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Note: The dot product of two vectors is a number

Ex: In $\mathbb{R}^2$, let $\vec{v} = \langle 3, 1 \rangle$ and $\vec{w} = \langle 4, -1 \rangle$.

Then,

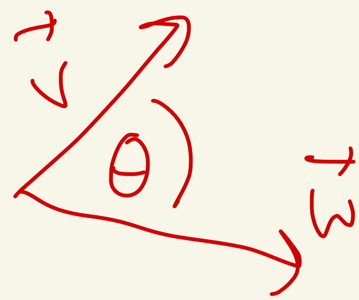
$$\vec{v} \cdot \vec{w} = \langle 3, 1 \rangle \cdot \langle 4, -1 \rangle$$

$$= (3)(4) + (1)(-1)$$

$$= 11$$

Calculus: In $\mathbb{R}^2$ and $\mathbb{R}^3$

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}$$



where θ is the angle between $\vec{v}$ and $\vec{w}$

$\boxed{\text{Ex:}}$

In $\mathbb{R}^6$ we have

$$\langle -1, 0, 2, 3, 4, 0 \rangle \cdot \langle 1, 1, 0, 0, -1, 2 \rangle$$

$$= (-1)(1) + (0)(1) + (2)(0) + (3)(0) + (4)(-1) + (0)(2)$$

$$= -1 + 0 + 0 + 0 - 4 + 0$$

$$= -5$$

$\boxed{\text{Properties of the dot product}}$

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$ and α be a scalar in $\mathbb{R}$.

Then:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $\alpha(\vec{u} \cdot \vec{v}) = (\alpha \vec{u}) \cdot \vec{v}$
   $= \vec{u} \cdot (\alpha \vec{v})$

Let's prove 2 when $n=3$

Let $\vec{u}, \vec{v}, \vec{w}$ be in $\mathbb{R}^3$.

We want to show that

$$\boxed{\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}}$$

We know that

$$\vec{u} = \langle a_1, a_2, a_3 \rangle$$

$$\vec{v} = \langle b_1, b_2, b_3 \rangle$$

and $\vec{w} = \langle c_1, c_2, c_3 \rangle$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$.

We have that

$$\vec{u}\cdot(\vec{v}+\vec{w})$$

$$= \langle a_1, a_2, a_3 \rangle \cdot \left(\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle \right)$$

$$= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$$

$$= a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3)$$

$$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \quad (*)$$

Also,

$$\vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$$

$$= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle + \langle a_1, a_2, a_3 \rangle \cdot \langle c_1, c_2, c_3 \rangle$$

$$\left.\begin{aligned} &= a_1b_1 + a_2b_2 + a_3b_3 \\ &\quad + a_1c_1 + a_2c_2 + a_3c_3 \end{aligned}\right\} (**)$$

Since $(*) = (**)$ we have that

$$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$$

$\blacksquare$

HW 1 - Part 1

10 List 3 elements from the set

$$S = \{ c_1 \langle 1,1,1 \rangle + c_2 \langle 0,0,5 \rangle \mid c_1, c_2 \in \mathbb{R} \}$$

read: S consists of all vectors of the form $c_1 \langle 1,1,1 \rangle + c_2 \langle 0,0,5 \rangle$ where c_1, c_2 are real numbers

When $c_1 = 7$, $c_2 = 11$ we get

$$7 \cdot \langle 1,1,1 \rangle + 11 \cdot \langle 0,0,5 \rangle$$

$$= \langle 7,7,7 \rangle + \langle 0,0,55 \rangle$$

$$= \langle 7,7,62 \rangle$$

So, $\langle 7,7,62 \rangle$ is in S.

When $c_1 = 0$, $c_2 = 0$ we get

$$0 \cdot \langle 1, 1, 1 \rangle + 0 \cdot \langle 0, 0, 5 \rangle$$

$$= \langle 0, 0, 0 \rangle + \langle 0, 0, 0 \rangle$$

$$= \langle 0, 0, 0 \rangle$$

So, $\langle 0, 0, 0 \rangle$ is in S

When $c_1 = -1$, $c_2 = 0$ we get

$$(-1) \cdot \langle 1, 1, 1 \rangle + 0 \cdot \langle 0, 0, 5 \rangle$$

$$= \langle -1, -1, -1 \rangle + \langle 0, 0, 0 \rangle$$

$$= \langle -1, -1, -1 \rangle$$

So, $\langle -1, -1, -1 \rangle$ is in S

So,

$$S = \{\langle 7,7,6\rangle, \langle 0,0,0\rangle, \langle -1,-1,-1\rangle, \ldots\}$$

infinitely many more

Topic 2 - Matrices

<u>Def:</u> A <u>matrix</u> is a rectangular array of numbers. If M is a matrix and it has m rows and n columns then we say that M is an $\underbrace{m \times n}$ matrix.

read: "m by n"

Abstractly, we can write an $m \times n$ matrix like this:

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where a_{ij} is the entry in the ith row and jth column of M.

Ex: Let

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 10 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$A \text{ is } \underset{\text{2 rows}}{2} \times \underset{\text{2 columns}}{2}$$

$$\boxed{\begin{aligned} a_{11} &= 0 \\ a_{12} &= 1 \\ a_{21} &= 2 \\ a_{22} &= 10 \end{aligned}}$$

Ex: Let

$$B = \begin{pmatrix} 1 & 0 & -1 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \end{pmatrix}$$

B is a 1×4 matrix.

$$\boxed{b_{11} = 1, \; b_{12} = 0, \; b_{13} = -1, \; b_{14} = 3}$$

You could use commas:

$$B = (1, 0, -1, 3)$$

Note: Sometimes we want to think of a vector as a matrix.

Suppose $\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$ in $\mathbb{R}^n$

We can think of $\vec{v}$ as an $n \times 1$ matrix: $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

Or we can think of $\vec{v}$ as a $1 \times n$ matrix: $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$

Ex: $\vec{v} = \langle 1, 2, 3 \rangle$

$\underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{\vec{v} \text{ as a } 3\times 1 \text{ matrix}}$ $\qquad$ $\underbrace{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}}_{\vec{v} \text{ as a } 1\times 3 \text{ matrix}}$

Def: Let A and B be $m \times n$ matrices [A & B have the same size]

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

and

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

Define $A+B$ to be the following $m \times n$ matrix:

$$A+B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

Define $A-B$ to be the following $m \times n$ matrix:

$$A-B = \begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \cdots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \cdots & a_{2n}-b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \cdots & a_{mn}-b_{mn} \end{pmatrix}$$

If α is a scalar in $\mathbb{R}$, then define αA to be the following $m \times n$ matrix:

$$\alpha A = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix}$$