Math 2550-01

1/30/24

Topic 1 – Vectors

<u>Def:</u> Let $n \geq 1$ be an integer.

[So, n can be $1, 2, 3, 4, 5, \ldots$]

An <u>n-dimensional real vector</u> is a list of n numbers.

We use brackets $<$ and $>$ to denote vectors and commas to separate the numbers.

We use an arrow over a variable to denote a vector, such as $\vec{v}$.

Ex: Some 2-dimensional real vectors:

$$\langle 8, -\tfrac{1}{2} \rangle$$

$$\langle \pi, 1.37 \rangle$$

Ex: Some 3-dimensional vectors:

$$\langle 1, 1, 1 \rangle$$

$$\langle 5, 3, -1 \rangle$$

$$\langle -1, 3, 5 \rangle$$

different

order matters

Ex: Some 5-dimensional vectors:

$$\langle 0, 27, 1, 9, 5 \rangle$$

$$\langle 2, \tfrac{1}{2}, -10000, -5, 2 \rangle$$

Def: Let $n \geq 1$ be an integer.

[So, n can be 1, 2, 3, 4, 5, ...]

Define $\mathbb{R}^n$ to be the set of all n-dimensional real vectors.

That is,

$$\mathbb{R}^n = \underbrace{\left\{ \langle a_1, a_2, \ldots, a_n \rangle \;\middle|\; a_1, a_2, \ldots, a_n \in \mathbb{R} \right\}}$$

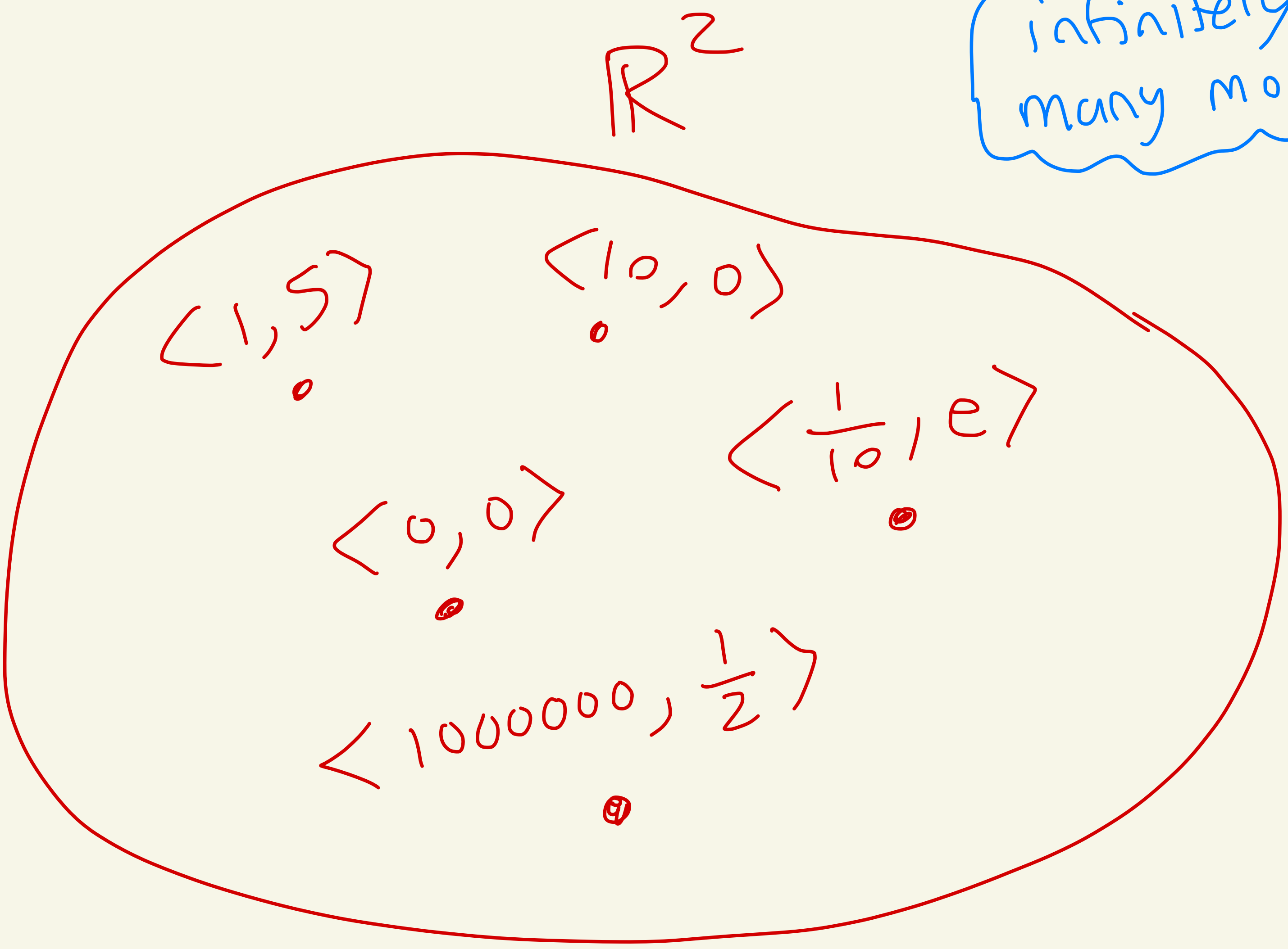
read: the set of all $\langle a_1, a_2, \ldots, a_n \rangle$ where $a_1, a_2, \ldots, a_n$ are real #s

Ex: $(n=2)$

$$\mathbb{R}^2 = \{ \langle a_1, a_2 \rangle \mid a_1, a_2 \in \mathbb{R} \}$$

$$= \{ \langle 1,5 \rangle, \langle 10,0 \rangle, \ldots \}$$

infinitely many more



Ex: $(n=5)$

$$\mathbb{R}^5 = \left\{ \langle a_1, a_2, a_3, a_4, a_5 \rangle \;\middle|\; \begin{array}{l} a_1, a_2, a_3, a_4, a_5 \\ \text{are real \#s} \end{array} \right\}$$

$$= \left\{ \langle 1, 2, 20, 3, -7 \rangle, \langle 1, 0, 0, 0, 2 \rangle, \langle 5, 4, 3, 2, 1 \rangle, \ldots \right\}$$

infinitely many more

Def: The **length** (or **norm** or **magnitude**) of the vector $\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$ in $\mathbb{R}^n$ is

$$\|\vec{v}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

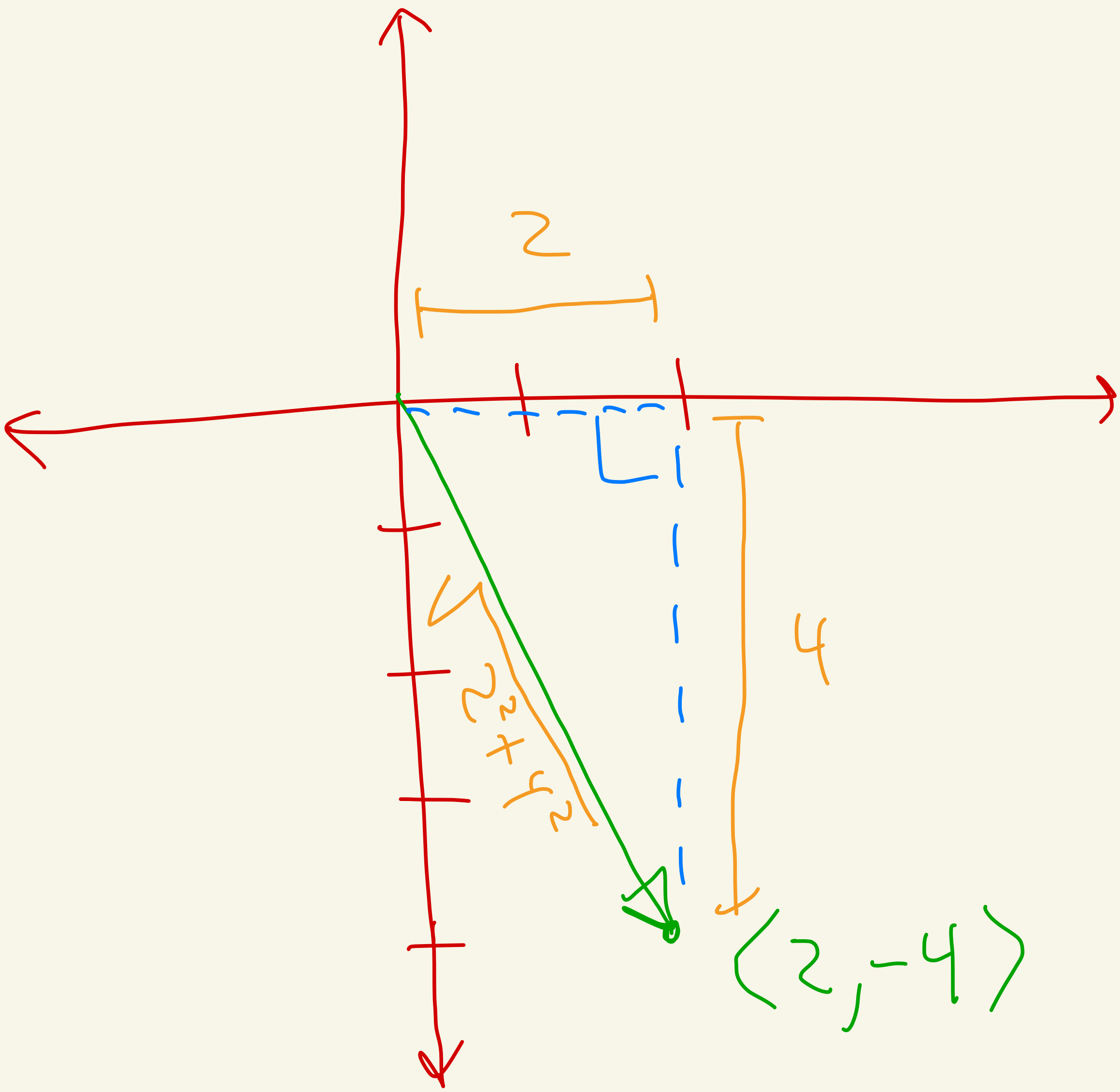
Some people use $|\vec{v}|$ instead of $\|\vec{v}\|$.

Ex: In $\mathbb{R}^2$, let $\vec{v} = \langle 2, -4 \rangle$

Then,

$$\|\vec{v}\| = \sqrt{2^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}$$

$$\approx 4.4721\ldots$$



Ex: In $\mathbb{R}^{10}$, let

$$\vec{w} = \langle 1, 0, 0, 0, -2, 3, 0, 0, 3, 4 \rangle.$$

Then,

$$\|\vec{w}\| = \sqrt{1^2 + 0^2 + 0^2 + 0^2 + (-2)^2 + 3^2 + 0^2 + 0^2 + 3^2 + 4^2}$$

$$= \sqrt{1 + 4 + 9 + 9 + 16}$$

$$= \sqrt{39}$$

$$\approx 6.2449\ldots$$

Operations on vectors

Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$ and let α and β be $\underbrace{\text{scalars}}_{\text{number}}$ in $\mathbb{R}$

some greek letters
α - alpha
β - beta
σ - sigma
ω - omega
δ - delta
γ - gamma

Suppose

$$\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$$

and

$$\vec{w} = \langle b_1, b_2, \ldots, b_n \rangle.$$

Define vector addition as

$$\vec{v} + \vec{w} = \langle a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n \rangle$$

Define vector subtraction as

$$\vec{v} - \vec{w} = \langle a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n \rangle$$

Define vector scaling as

$$\alpha \vec{v} = \langle \alpha a_1, \alpha a_2, \ldots, \alpha a_n \rangle$$

Ex: In $\mathbb{R}^2$, let $\vec{v} = \langle 2, -3 \rangle$ and $\vec{w} = \langle 1, -1 \rangle$.

Then:

$$\vec{v} + \vec{w} = \langle 2, -3 \rangle + \langle 1, -1 \rangle$$
$$= \langle 2+1, -3-1 \rangle = \boxed{\langle 3, -4 \rangle}$$

$$\vec{w} - \vec{v} = \langle 1, -1 \rangle - \langle 2, -3 \rangle$$
$$= \langle 1-2, -1-(-3) \rangle = \boxed{\langle -1, 2 \rangle}$$

$$\underbrace{(-5)}_{\alpha}\vec{v} = (-5)\langle 2, -3 \rangle$$
$$= \langle (-5)(2), (-5)(-3) \rangle$$
$$= \langle -10, 15 \rangle$$

Ex: In $\mathbb{R}^5$, we have

$$\langle 1, 0, -1, 2, \tfrac{1}{2} \rangle + \langle 0, -1, 3, -2, 5 \rangle$$

$$= \langle 1+0, 0+(-1), -1+3, 2+(-2), \tfrac{1}{2}+5 \rangle$$

$$= \langle 1, -1, 2, 0, 5.5 \rangle$$

$$3\langle 1, 2, -1, 2, 4 \rangle$$

$$= \langle 3, 6, -3, 6, 12 \rangle$$

$\boxed{\text{Notation:}}$ In $\mathbb{R}^n$, the zero vector is the vector containing all 0's.
It's notated by $\vec{0}$.

In $\mathbb{R}^2$, $\vec{0} = \langle 0, 0 \rangle$.
In $\mathbb{R}^3$, $\vec{0} = \langle 0, 0, 0 \rangle$.
In $\mathbb{R}^4$, $\vec{0} = \langle 0, 0, 0, 0 \rangle$.

And so on.

Properties of vectors:

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$

and α, β be $\underbrace{\text{scalars}}_{\text{numbers}}$ in $\mathbb{R}$.

Then:

① $\vec{u} + \vec{w} = \vec{w} + \vec{u}$ ← (commutivity)

② $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ ← (associativity)

③ $\alpha(\beta \vec{u}) = (\alpha\beta)\vec{u}$ ← $\boxed{2(5\vec{u}) = 10\vec{u}}$

④ $(\alpha + \beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$ ← $\boxed{5\vec{u} = 3\vec{u} + 2\vec{u}}$

⑤ $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$ ← $\boxed{2(\vec{u} + \vec{v}) = 2\vec{u} + 2\vec{v}}$

⑥ $\vec{u} + \vec{0} = \vec{u}$ and $\vec{0} + \vec{u} = \vec{u}$

⑦ $\vec{u} + (-\vec{u}) = \vec{0}$

$(-\vec{u}) + \vec{u} = \vec{0}$

$\boxed{\text{proof of (3) when } n = 2:}$

Let $\vec{u}$ be in $\mathbb{R}^2$ and α, β be in $\mathbb{R}$.

Then, $\vec{u} = \langle a_1, a_2 \rangle$ where a_1, a_2 are real numbers.

Then,

$$\begin{aligned} \alpha(\beta \vec{u}) &= \alpha(\beta \langle a_1, a_2 \rangle) \\ &= \alpha \langle \beta a_1, \beta a_2 \rangle \\ &= \langle \alpha\beta a_1, \alpha\beta a_2 \rangle \\ &= (\alpha\beta) \langle a_1, a_2 \rangle \\ &= (\alpha\beta) \vec{u} \end{aligned}$$

$\blacksquare$

end of proof