2550

HW 9

Solutions

1 (a) $A = \begin{pmatrix} 1 & 3 \\ 4 & -6 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} -2 \\ 10 \end{pmatrix}$

We want to see if $\vec{b}$ is in the columnspace of A, So we want to see if we can solve

$$\boxed{\underbrace{\begin{pmatrix} -2 \\ 10 \end{pmatrix}}_{\vec{b}} = \underbrace{x_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -6 \end{pmatrix}}_{\text{linear combo of columns of } A}} \quad (*)$$

for some scalars x_1, x_2.

Notice that we can rewrite this equation as

$$\begin{pmatrix} -2 \\ 10 \end{pmatrix} = \begin{pmatrix} x_1 \\ 4x_1 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ -6x_2 \end{pmatrix}$$

which is equivalent to $\begin{pmatrix} -2 \\ 10 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ 4x_1 - 6x_2 \end{pmatrix}$

which is equivalent to $\begin{pmatrix} -2 \\ 10 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

We have turned the problem into a question of: can we solve the above $\vec{b} = A\vec{x}$ equation.

Let's see if we can solve it.

$$\left(\begin{array}{cc|c} 1 & 3 & -2 \\ 4 & -6 & 10 \end{array}\right) \xrightarrow{-4R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 3 & -2 \\ 0 & -18 & 18 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{18}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 3 & -2 \\ 0 & 1 & -1 \end{array}\right)$$

Now we try to solve:

$$\boxed{\begin{aligned} x_1 + 3x_2 &= -2 \\ x_2 &= -1 \end{aligned}} \Rightarrow \boxed{\begin{aligned} x_1 &= -2-3x_2 \\ x_2 &= -1 \end{aligned}} \Rightarrow \boxed{\begin{aligned} x_1 &= -2+3 = 1 \\ x_2 &= -1 \end{aligned}}$$

So we <u>can</u> solve (*) and we get

$$\boxed{\vec{b} = \begin{pmatrix} -2 \\ 10 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} - 1 \cdot \begin{pmatrix} 3 \\ -6 \end{pmatrix}}$$

So, $\vec{b}$ is in the column space of A because it can be written as a linear combination of the columns of A.

$\boxed{1(b)}$ $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

We want to see if $\vec{b}$ is in the columnspace of A, So we want to see if we can solve

$$\boxed{\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}} \quad (*)$$

for some scalars x_1, x_2, x_3.

We can rewrite this equation as

$$\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \\ 2x_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ 0 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2x_3 \\ x_3 \\ 3x_3 \end{pmatrix}$$

Which is equivalent to

$$\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + 2x_3 \\ x_1 \qquad + x_3 \\ 2x_1 + x_2 + 3x_3 \end{pmatrix}$$

Which is equivalent to

$$\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

So we have converted this problem into an equation of the form $\vec{b} = A\vec{x}$

Let's see if its solvable

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \end{array}\right) \xrightarrow[-2R_1+R_2\to R_2]{-R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 4 \end{array}\right)$$

$$\xrightarrow{-R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 4 \end{array}\right) \xrightarrow{R_2+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

This gives

$$\boxed{\begin{aligned} x_1+x_2+2x_3 &= -1 \\ x_2+x_3 &= -1 \\ 0 &= 3 \end{aligned}}$$

There are no solutions to this system since we have $0=3$.

Thus, there are no solutions to (*) on the previous page and $\vec{b}$ is <u>not</u> in the column space of A.

$\boxed{1(c)}$ $A = \begin{pmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$

We solve in the same way as 1(a) & 1(b).

Can we solve

$$\boxed{\vec{b} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 9 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \quad (*)$$

for x_1, x_2, x_3?

This equation becomes

$$\vec{b} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 + x_3 \\ 9x_1 + 3x_2 + x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Let's try to solve this system.

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 9 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right) \xrightarrow[-R_1+R_3\to R_3]{-9R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 12 & -8 & -44 \\ 0 & 2 & 0 & -6 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 2 & 0 & -6 \\ 0 & 12 & -8 & -44 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2\to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 12 & -8 & -44 \end{array}\right)$$

$$\xrightarrow{-12R_2+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & -8 & -8 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{8}R_3\to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\boxed{\begin{aligned} x_1 - x_2 + x_3 &= 5 \\ x_2 &= -3 \\ x_3 &= 1 \end{aligned}} \Rightarrow \boxed{\begin{aligned} x_3 &= 1 \\ x_2 &= -3 \\ x_1 &= 5 + x_2 - x_3 \\ &= 5 - 3 - 1 = 1 \end{aligned}}$$

So, yes $\vec{b}$ is in the column space of A and (*) becomes

$$\boxed{\vec{b} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 9 \\ 1 \end{pmatrix} - 3 \cdot \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

$\boxed{2(a)}$ $A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix}$

(i) We find a basis for the nullspace. Recall that the nullspace of A is all the solutions to $A\vec{x} = \vec{0}$ that is the solutions to

$$\begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which is equivalent to solving

$$\begin{pmatrix} x_1 - x_2 + 3x_3 \\ 5x_1 - 4x_2 - 4x_3 \\ 7x_1 - 6x_2 + 2x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which is the system

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ 5x_1 - 4x_2 - 4x_3 &= 0 \\ 7x_1 - 6x_2 + 2x_3 &= 0 \end{aligned}$$

Solving we have

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 5 & -4 & -4 & 0 \\ 7 & -6 & 2 & 0 \end{array}\right) \xrightarrow[-7R_1+R_3 \to R_3]{-5R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 1 & -19 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So,

$$\boxed{\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ x_2 - 19x_3 &= 0 \\ 0 &= 0 \end{aligned}} \Longrightarrow \boxed{\begin{aligned} x_3 &= t \\ x_2 &= 19x_3 = 19t \\ x_1 &= x_2 - 3x_3 \\ &= 19t - 3t \\ &= 16t \end{aligned}}$$

So the nullspace of A is

$$\underbrace{N(A)}_{\text{notation for nullspace of A}} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0} \right\}$$

$$= \left\{ \begin{pmatrix} 16t \\ 19t \\ t \end{pmatrix} \middle| t \text{ is in } \mathbb{R} \right\} =$$

$$= \left\{ t \begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix} \;\middle|\; t \text{ in } \mathbb{R} \right\}$$

$$= \text{span}\left(\left\{ \begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix} \right\} \right)$$

So, $\begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix}$ spans the nullspace of A.

This vector is lin. ind. since if

$$c_1 \begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ then } \begin{pmatrix} 16c_1 \\ 19c_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and so $c_1 = 0$ [by the bottom equation]

Thus, a basis for the nullspace is $\begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix}$.

(ii) The nullity of A is the dimension of the nullspace of A. Since the nullspace of A has a basis of size 1, the nullity of A is 1.

(iii) Now for the column space.

We saw in part (i) that the row echelon form of

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix} \text{ is } \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{pmatrix}.$$

Circle the leading 1's in the row-echelon form of A.

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{pmatrix}$$

This corresponds to column 1 and column 2

So columns 1 and 2 of A form a basis for the column space of A.

That is, $\left\{ \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ -6 \end{pmatrix} \right\}$ form a basis for the column space of A.

(iv) The rank of A is the dimension of the column space of A which is the number of elements in a basis for the column space of A. By (iii) the column space has dimension 2.

(v) A is 3×3 $\left[m \times n \text{ where } \begin{matrix} m=3 \\ n=3 \end{matrix} \right]$

The rank-nullity thm says that

$$\text{rank}(A) + \text{nullity}(A) = n$$

$\boxed{n = \#\text{ columns of } A}$

In this problem we have that this equation becomes

$$2 + 1 = 3$$

which is true. So, we have verified the rank-nullity thm for this matrix.

$\boxed{2(b)}$ $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

(i) The nullspace of A consists of all $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ where $A\vec{x} = \vec{0}$, that is all $\vec{x}$ where

$$\begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which is the same as solving

$$\begin{pmatrix} 2x_1 & -x_3 \\ 4x_1 & -2x_3 \\ & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} 2x_1 - x_3 &= 0 \\ 4x_1 - 2x_3 &= 0 \\ 0 &= 0 \end{aligned}}$$

Let's solve this system

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 4 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 4 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-4R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ which becomes

$$\boxed{\begin{aligned} x_1 \quad - \tfrac{1}{2} x_3 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}}$$

leading variable: x_1

free variables: x_2, x_3

So,
$$\boxed{\begin{aligned} x_1 &= \tfrac{1}{2} x_3 = \tfrac{1}{2} u \\ x_2 &= t \\ x_3 &= u \end{aligned} \quad \text{where } u, t \text{ are any real numbers}}$$

So, the nullspace of A is

$$\underbrace{N(A)}_{\text{notation for nullspace of } A} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \,\middle|\, A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} \frac{1}{2}u \\ t \\ u \end{pmatrix} \,\middle|\, t, u \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} \frac{1}{2}u \\ 0 \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} \,\middle|\, t, u \in \mathbb{R} \right\}$$

$$= \left\{ u\begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \,\middle|\, t, u \in \mathbb{R} \right\} =$$

$$= \text{span}\left(\left\{\begin{pmatrix}1/2\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}\right)$$

So, $\begin{pmatrix}1/2\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ span the nullspace of A.

Let's show they are linearly independent.

Suppose

$$c_1\begin{pmatrix}1/2\\0\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Then,

$$\begin{pmatrix}\frac{1}{2}c_1\\c_2\\c_1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

So, $c_1 = 0$ and $c_2 = 0$ from the bottom two equations.

Thus, a basis for the nullspace is $\left\{\begin{pmatrix}1/2\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$

(ii) Since the nullspace of A has a basis with two vectors, it has dimension two.

So, nullity $(A) = 2$.

(iii) If one row reduces

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$ as in part (i)

then one gets $\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\underbrace{\text{for the row-echelon form of } A.}$

Circling the leading 1's gives:

$$\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The leading 1 lives in column 1 of the row-echelon form of A.

So, column 1 of A is a basis for the column space of A.

That is, $\left\{ \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \right\}$ is a basis for the column space of A.

(iv) By part (iii) the column space has dimension 1 [a basis for the column space consists of one vector].

So, the rank of A is 1.

(v) A is $m \times n = 3 \times 3$.

The rank-nullity theorem says

$$\text{rank}(A) + \text{nullity}(A) = n$$

$n = \#$ columns of A

which for this example becomes

$$1 + 2 = 3$$

which is true. So we have verified the rank-nullity theorem for this matrix.

$\boxed{2(c)}$ $A = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix}$

(i) Same procedure as 2(a) and 2(b) solutions. We want to solve

$$\begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let's do it,

$$\left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 3 & 2 & 2 & 0 \end{array}\right) \xrightarrow[R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & -7 & -7 & -4 & 0 \\ 0 & 7 & 7 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{7}R_2\to R_2} \left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & 1 & 1 & 4/7 & 0 \\ 0 & 7 & 7 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{-7R_2+R_3\to R_3} \left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & 1 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

So we get:

$$\boxed{\begin{aligned} x_1 + 4x_2 + 5x_3 + 2x_4 &= 0 \\ x_2 + x_3 + \tfrac{4}{7}x_4 &= 0 \\ 0 &= 0 \end{aligned}}$$

So, $\boxed{\begin{array}{l} x_1 = -4x_2 - 5x_3 - 2x_4 \\ x_2 = -x_3 - \frac{4}{7}x_4 \end{array}}$

x_1, x_2 are leading variables

x_3, x_4 are free variables

$$x_3 = t$$
$$x_4 = u$$
$$x_2 = -t - \frac{4}{7}u$$
$$x_1 = -4\left(-t - \frac{4}{7}u\right) - 5t - 2u$$
$$= -t + \frac{2}{7}u$$

t, u are any real numbers

So the nullspace of A is

$$N(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| \; A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} -t + \frac{2}{7}u \\ -t - \frac{4}{7}u \\ t \\ u \end{pmatrix} \middle| \; u, t \text{ are real numbers} \right\} =$$

$$= \left\{ \begin{pmatrix} -t \\ -t \\ t \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{2}{7}u \\ -\frac{4}{7}u \\ 0 \\ u \end{pmatrix} \;\middle|\; t, u \text{ are real numbers} \right\}$$

$$= \left\{ t\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + u\begin{pmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; t, u \text{ are real numbers} \right\}$$

$$= \operatorname{Span}\left(\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{pmatrix} \right\} \right).$$

Let's check that these two vectors are linearly independent.

Suppose

$$c_1 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then, $\begin{pmatrix} -c_1 + 2/7\, c_2 \\ -c_1 - 4/7\, c_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ (pg 21)

The bottom two equations give that $c_1 = c_2 = 0$.

Thus $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for the nullspace of A.

(ii) The nullspace of A has a basis with two elements, hence it has dimension two. So, the nullity of A is 2.

(iii) Part (i) shows row-reducing

$$A = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix}$$ leads

to the row-reduced form

$$\begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & 4/7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ where the leading ones are circled.

These leading $\underline{1}$'s are in columns one and two of the row-reduced form of A. Thus, columns one and two are a basis for the column space of A. That is a basis is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} \right\}.$$

(iv) By part (iii) a basis for the column space of A has two elements, so the column space has dimension two.

Hence the rank of A is 2.

(v) A is $m \times n = 3 \times 4$.

The rank nullity theorem says

$$\text{nullity}(A) + \text{rank}(A) = n$$

$n = \#$ columns of A

This formula becomes

$$2 + 2 = 4$$

which is true.

So we have verified that the rank-nullity theorem is true for this matrix.

pg 24

$\boxed{3}$ We are given that A is $m \times n = 4 \times 5$ and that the nullity of A is 3.

The rank-nullity theorem tells us that

$$\text{rank}(A) + \text{nullity}(A) = n$$

which becomes

$$\text{rank}(A) + 3 = 5$$

So,

$$\text{rank}(A) = 2.$$

(4) Suppose A is $m \times n$.

We are given that a basis for the column space of A is
$\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \\ 8 \\ 7 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 1 \\ -9 \\ 6 \end{pmatrix} \right\}$. Thus, the column space has dimension 2. So, $\operatorname{rank}(A) = 2$.

We are also given that the nullspace has a basis of size 2. Thus, $\operatorname{nullity}(A) = 2$.

By the rank-nullity theorem,

$$n = \operatorname{rank}(A) + \operatorname{nullity}(A)$$

$n =$ # columns of A

So, the number of columns of A is

$$n = 2 + 2 = 4$$

$\boxed{5(a)}$ Let $\vec{v}_1 = \langle 2, -1 \rangle$
$\vec{v}_2 = \langle 5, -7 \rangle$
$\vec{v}_3 = \langle 1, 1 \rangle$

$\boxed{(i)}$

To find a subset of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ that is a basis for $\operatorname{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$ we put the vectors into a matrix as columns.

Let $A = \begin{pmatrix} 2 & 5 & 1 \\ -1 & -7 & 1 \end{pmatrix}$

So we are looking for a basis for the column space of A.

We need to row-reduce A.

$$\begin{pmatrix} 2 & 5 & 1 \\ -1 & -7 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & -7 & 1 \\ 2 & 5 & 1 \end{pmatrix}$$

$$\xrightarrow{-R_1 \to R_1} \begin{pmatrix} 1 & 7 & -1 \\ 2 & 5 & 1 \end{pmatrix} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{pmatrix} 1 & 7 & -1 \\ 0 & -9 & 3 \end{pmatrix} \to$$

$\xrightarrow{-\frac{1}{9}R_2 \to R_2} \begin{pmatrix} 1 & 7 & -1 \\ 0 & 1 & -1/3 \end{pmatrix}$

The row-echelon form of $A = \begin{pmatrix} 2 & 5 & 1 \\ -1 & -7 & 1 \end{pmatrix}$

is $\begin{pmatrix} 1 & 7 & -1 \\ 0 & 1 & -1/3 \end{pmatrix}$ where I've circled

the leading 1's. They are in columns one and two. So, columns one and two of A are a basis for the column space of A.

That is, $\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ -7 \end{pmatrix} \right\}$ is a basis for the column space of A.

So, $\operatorname{span}(\{\langle 2,-1\rangle, \langle 5,-7\rangle, \langle 1,1\rangle\})$

$= \operatorname{span}(\{\langle 2,-1\rangle, \langle 5,-7\rangle\})$

with basis $\{\langle 2,-1\rangle, \langle 5,-7\rangle\}$.

$\boxed{(ii)}$

Now we write the extra vector $\vec{v}_3$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$. Lets solve

$$\underbrace{\langle 1,1\rangle}_{\vec{v}_3} = c_1\underbrace{\langle 2,-1\rangle}_{\vec{v}_1} + c_2\underbrace{\langle 5,-7\rangle}_{\vec{v}_2}$$

This becomes $\langle 1,1\rangle = \langle 2c_1+5c_2, -c_1-7c_2\rangle$.

So,

$$\boxed{\begin{aligned} 2c_1+5c_2&=1 \\ -c_1-7c_2&=1 \end{aligned}}$$

Solving this we get

$$\left(\begin{array}{cc|c} 2 & 5 & 1 \\ -1 & -7 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} -1 & -7 & 1 \\ 2 & 5 & 1 \end{array}\right)$$

$$\xrightarrow{-R_1 \to R_1} \left(\begin{array}{cc|c} 1 & 7 & -1 \\ 2 & 5 & 1 \end{array}\right) \xrightarrow{-2R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 7 & -1 \\ 0 & -9 & 3 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{9}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 7 & -1 \\ 0 & 1 & -1/3 \end{array}\right)$$

which gives $\boxed{\begin{aligned} c_1 + 7c_2 &= -1 \\ c_2 &= -1/3 \end{aligned}}$ or $\boxed{\begin{aligned} c_2 &= -1/3 \\ c_1 &= -1-7c_2 \\ &= -1-7(-1/3) = \frac{4}{3} \end{aligned}}$

So,

$$\vec{v}_3 = \langle 1,1 \rangle = \frac{4}{3}\langle 2,-1 \rangle - \frac{1}{3}\langle 5,-7 \rangle$$
$$= \frac{4}{3}\vec{v}_1 - \frac{1}{3}\vec{v}_2$$

$\boxed{5(b)}$ Let $\vec{v}_1 = \langle 1,0,1 \rangle$
$\vec{v}_2 = \langle 0,1,2 \rangle$
$\vec{v}_3 = \langle 1,1,1 \rangle$

$\boxed{(i)}$ To find a subset of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ that is a basis for $\text{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$ we put the vectors into a matrix as columns. Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

We now row reduce A to find a basis for the column space of A.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{-R_1+R_3 \to R_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow{-2R_2+R_3 \to R_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_3 \to R_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

So the row-echelon form of $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ where I've circled the leading 1's. The leading 1's are in columns one, two, and three. So, the corresponding columns one, two, and three of A are a basis for the column space of A. $\blacksquare$

That is, a basis for the column space of A is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

So, a basis for the span of $\vec{v}_1 = \langle 1,0,1 \rangle$, $\vec{v}_2 = \langle 0,1,2 \rangle$, $\vec{v}_3 = \langle 1,1,1 \rangle$ $\underline{\underline{\text{is}}}$ $\vec{v}_1 = \langle 1,0,1 \rangle$, $\vec{v}_2 = \langle 0,1,2 \rangle$, $\vec{v}_3 = \langle 1,1,1 \rangle$.

$\boxed{\text{(ii)}}$ All of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are a basis for the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3$. So there is nothing to do here.

$\boxed{6}$ We are given that A is an $m \times n = 3 \times 3$ matrix and that $\text{nullity}(A) = 0$.

pg 32

By the rank-nullity theorem

$$\text{rank}(A) + \text{nullity}(A) = n$$

which becomes

$$\text{rank}(A) + 0 = 3.$$

Thus, $\text{rank}(A) = 3$.

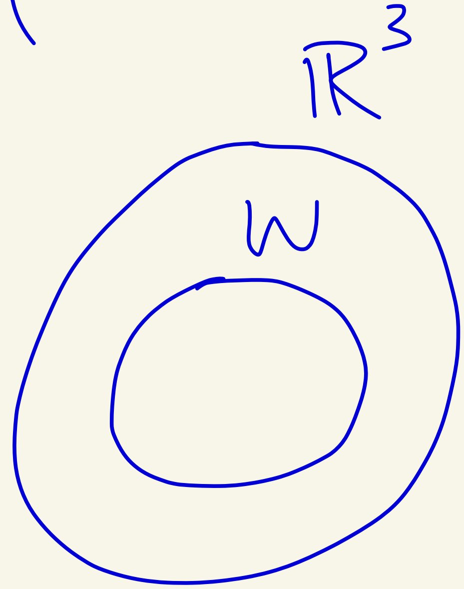
Since A is 3×3, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

The column space of A is

$$W = \text{span}\left(\left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \right\} \right)$$

which has dimension 3 since $\text{rank}(A) = 3$.

So, W = column space of A is of dimension 3 and it lives in the 3 dimensional space $\mathbb{R}^3$.



By a theorem in class this implies that

$W = \mathbb{R}^3$.

Thus, every vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$

is in W which is the column space of A.

So, every vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ is

in the span of the columns of A.