Topic 3 —
Systems of
linear equations

HW 3 Topic - Systems of linear equations

Def: A linear equation in the n variables $x_1, x_2, \dots, x_n$ is an equation of the form

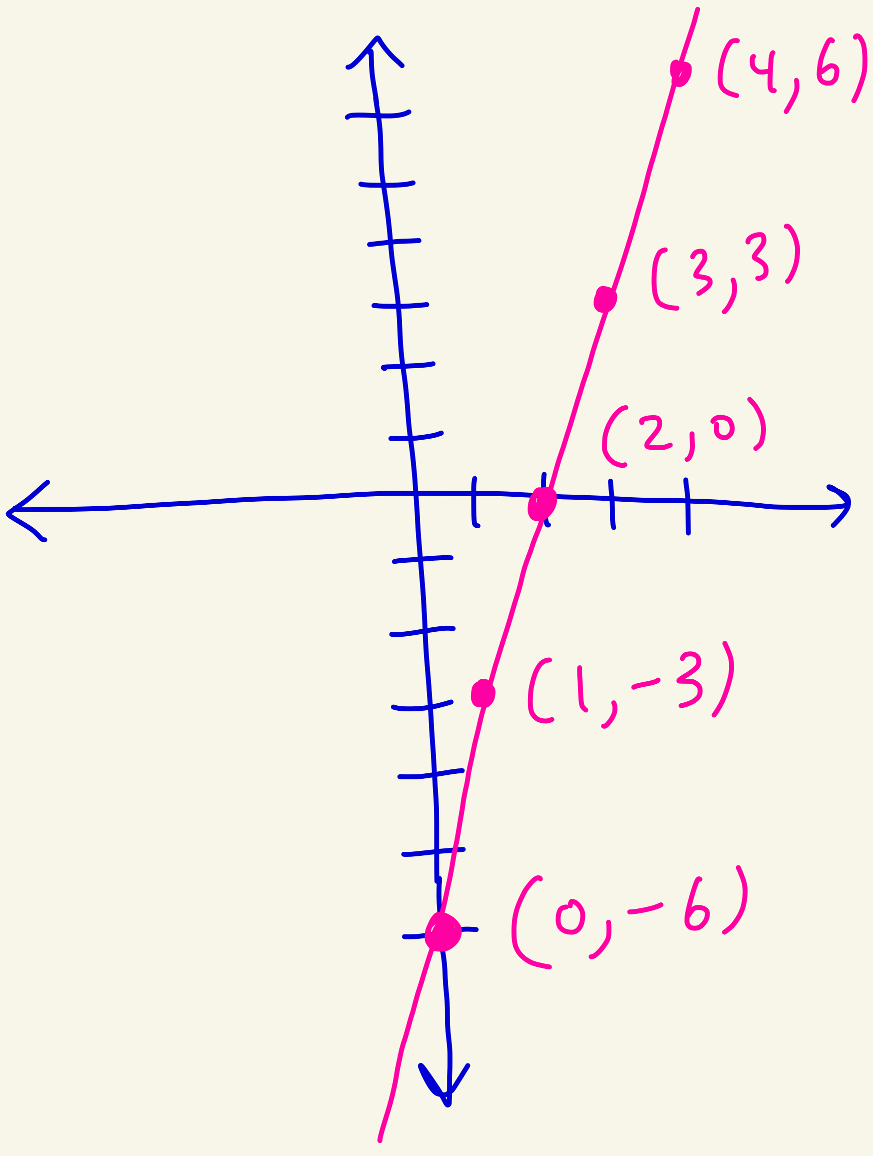
$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b \quad (*)$$

where $a_1, a_2, \dots, a_n, b$ are constant real numbers.

The solution space of the above equation $(*)$ consists of the set of all $(x_1, x_2, \dots, x_n)$ that solves the equation.

Ex: $3x - y = 6$

is a linear equation in two variables x, y.



pink line is the solution space

Another way to describe the solution space to $3x-y=6$ is as the following set:

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \underbrace{3x-y=6}_{y=3x-6} \text{ and } x,y\in\mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ 3x-6 \end{pmatrix} \;\middle|\; \underbrace{x \text{ is a real number}}_{x\in\mathbb{R}} \right\}$$

$$= \left\{ \underbrace{\begin{pmatrix} 0 \\ -6 \end{pmatrix}}_{x=0}, \underbrace{\begin{pmatrix} -1 \\ -9 \end{pmatrix}}_{x=-1}, \underbrace{\begin{pmatrix} \pi \\ 3\pi-6 \end{pmatrix}}_{x=\pi}, \ldots \right\}$$

infinitely many more

(another way:)

$$= \left\{ \begin{pmatrix} x \\ 3x \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \end{pmatrix} \;\middle|\; x\in\mathbb{R} \right\}$$

$$= \left\{ x\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \end{pmatrix} \;\middle|\; x\in\mathbb{R} \right\}$$

Ex: Some more linear equations:

$$\sqrt{2}\, w - \frac{1}{2} z = 0$$

$$10x + 1000y + \frac{1}{2} z - \frac{1}{3} w = 5$$

$$x_1 + 10x_2 - 30x_3 = \frac{1}{2}$$

Ex: Some non-linear equations:

$$2y + x^2 = 7$$

$$5\cos(x) + 37x = 2$$

Def: A system of m linear equations in the n unknowns $x_1, x_2, \ldots, x_n$ is a set of m equations of the form

$$\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}} \tag{$*$}$$

where the a_{ij} are constant real numbers

The augmented matrix for $(*)$ is

$$\begin{array}{c} \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_2 \end{array}\right) \\ \begin{array}{cccc} x_1 \text{ column} & x_2 \text{ column} & x_n \text{ column} & \end{array} \end{array}$$

represents the $=$ sign

The <u>solution space</u> of the system (*) consists of all the $(x_1, x_2, \ldots, x_n)$ that simultaneously solve all m equations.

That is, the common solutions to all m equations.

Ex:

$$\left.\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned}\right]$$

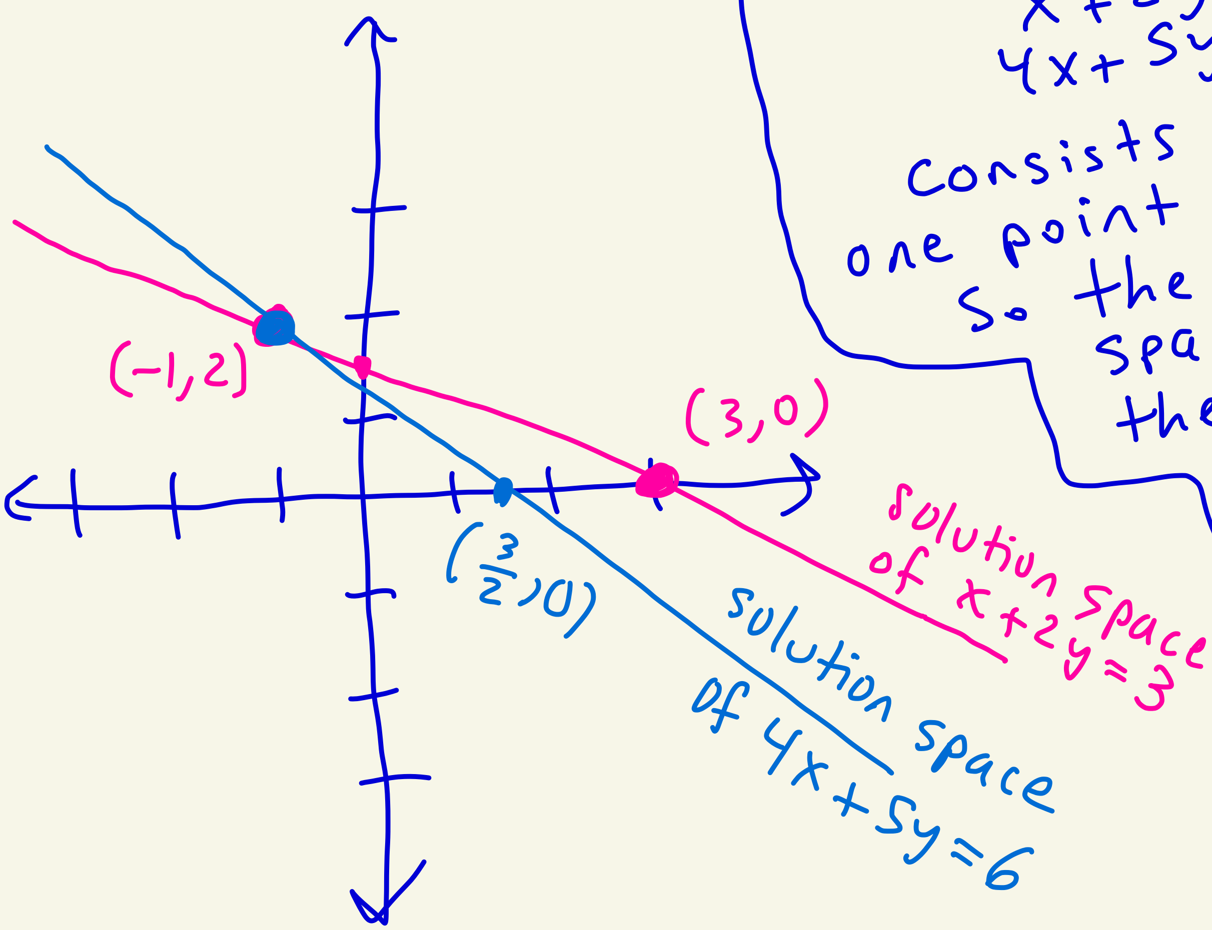
system of $m=2$ linear equations and $n=2$ unknowns

Augmented matrix:

$$\begin{array}{ccc} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right) \\ \end{array}$$

$$\begin{array}{ccc} x & y & = \text{ sign} \end{array}$$

Solution space:



The solution space for
$x + 2y = 3$
$4x + 5y = 6$
consists of just one point $(x,y) = (-1,2)$. So the solution space is the set $\{(-1,2)\}$

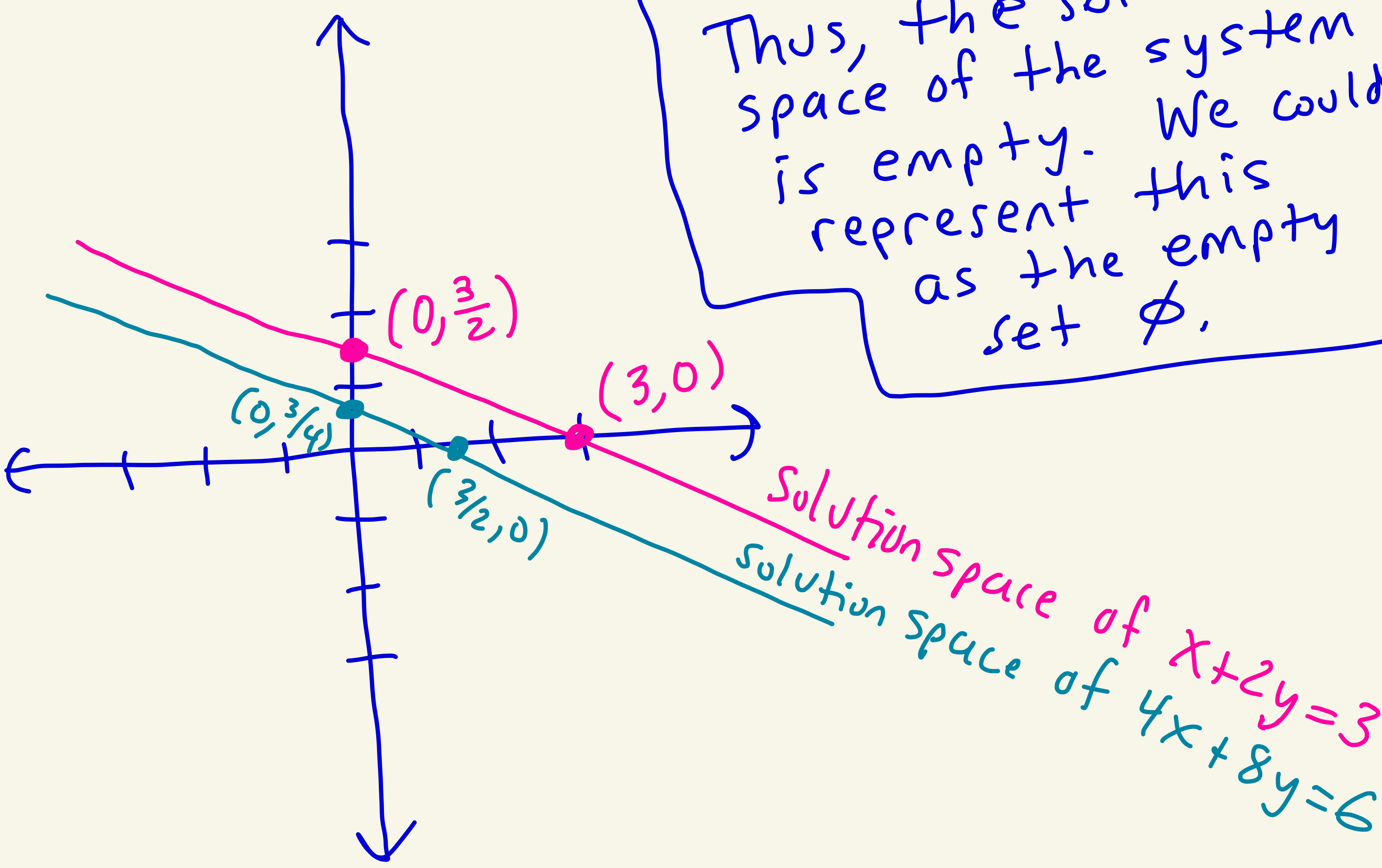
Ex: Consider the system

$$\left.\begin{aligned} x+2y&=3 \\ 4x+8y&=6 \end{aligned}\right]$$ system of $m=2$ linear equations and $n=2$ unknowns

Augmented matrix $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 8 & 6 \end{array}\right)$

Solution space

These lines are parallel. So there is no common solution to the system. Thus, the solution space of the system is empty. We could represent this as the empty set ϕ.



Ex:

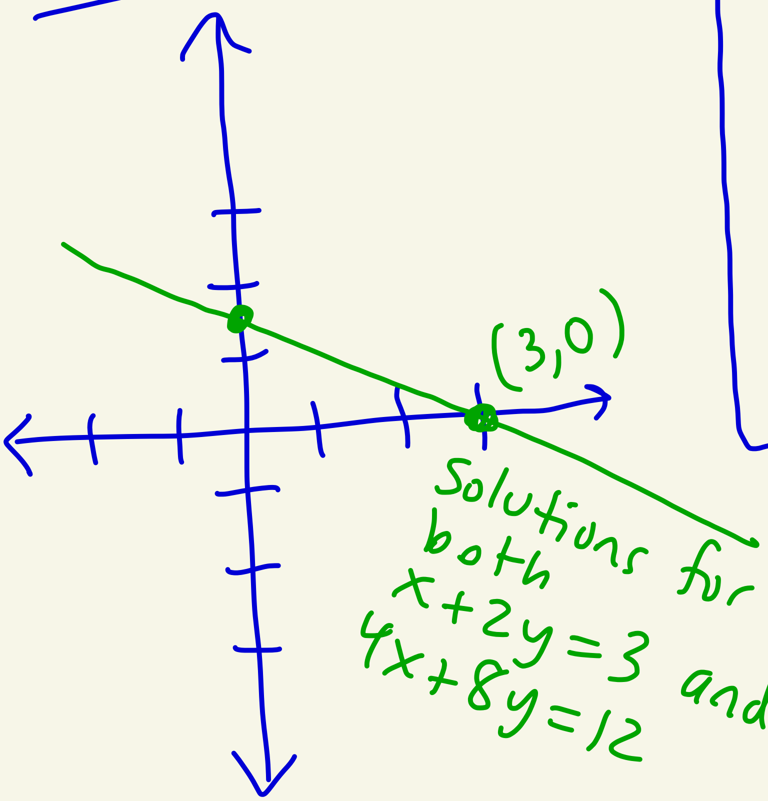
$$x + 2y = 3$$
$$4x + 8y = 12$$

system of $m=2$ linear equations and $n=2$ unknowns

Augmented matrix

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 8 & 12 \end{array}\right)$$

Solution space



It's the same line twice.

Solution space for the system is the line $x+2y=3$, or

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \begin{array}{c} x+2y=3 \\ x, y \in \mathbb{R} \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} 3-2y \\ y \end{pmatrix} \,\middle|\, y \in \mathbb{R} \right\}$$

$$= \left\{ \underbrace{\begin{pmatrix} 3 \\ 0 \end{pmatrix}}_{y=0}, \underbrace{\begin{pmatrix} 5 \\ -1 \end{pmatrix}}_{y=-1}, \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{y=1}, \ldots \right\}$$

infinitely many more

Ex:

$$\left.\begin{aligned} x + y + 2z &= 9 \\ 2x - 3z &= 1 \\ -x + 6y - 5z &= 0 \end{aligned}\right]$$

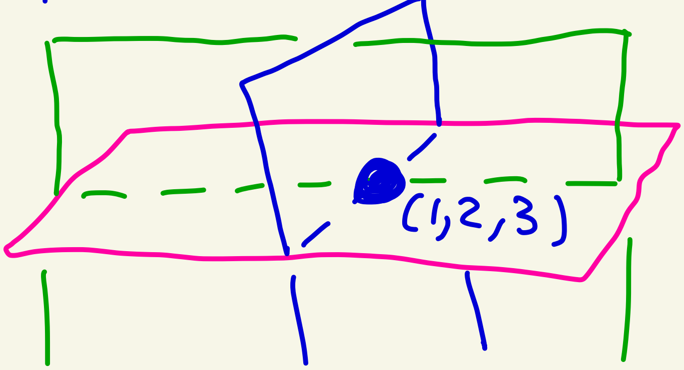
system of $m=3$ equations and $n=3$ unknowns

Augmented Matrix

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 0 & -3 & 1 \\ -1 & 6 & -5 & 0 \end{array}\right) \\ \begin{array}{ccc} x \text{ column} & y \text{ column} & z \text{ column} \end{array} \end{array}$$

If you drew the picture it would be three planes in 3d. Later we will see that the solution space (which is where the 3 planes intersect) is just one point

$$(x,y,z) = (1,2,3)$$



(made up drawing)

Ex:

$$\left.\begin{aligned} x + 4y - 2w + z &= 1 \\ 2x + w &= 3 \\ 14y - 12w + 7z &= 0 \end{aligned}\right\}$$

system with $m=3$ equations and $n=4$ unknowns

Augmented matrix

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 4 & -2 & 1 & 1 \\ 2 & 0 & 1 & 0 & 3 \\ 0 & 14 & -12 & 7 & 0 \end{array}\right) \\ \begin{array}{cccc} \underbrace{\quad}_{x} & \underbrace{\quad}_{y} & \underbrace{\quad}_{w} & \underbrace{\quad}_{z} \end{array} \end{array}$$

Now we want to learn how to solve systems of linear equations. We will learn a method called Gaussian elimination or row reduction.

We need some definitions and terminology first

Def: Given a system of linear equations there are three operations that we call elementary row operations

They are:

1. Multiply one of the rows/equations by a non-zero constant.
2. Interchange two rows/equations.
3. Add a multiple of one row/equation to a different row/equation.

Ex: (Multiply a row/equation by a non-zero constant)

Equation viewpoint

$$\boxed{\begin{aligned} 3x - y + z &= 1 \\ 5x \quad + 2z &= 2 \\ x + y + z &= -1 \end{aligned}} \xrightarrow{3R_2 \to R_2} \boxed{\begin{aligned} 3x - y + z &= 1 \\ 15x \quad + 6z &= 6 \\ x + y + z &= -1 \end{aligned}}$$

Augmented matrix viewpoint

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 5 & 0 & 2 & 2 \\ 1 & 1 & 1 & -1 \end{array}\right) \xrightarrow{3R_2 \to R_2} \left(\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 15 & 0 & 6 & 6 \\ 1 & 1 & 1 & -1 \end{array}\right)$$

Ex: (Interchanging two rows/equations)

Equation viewpoint

$$\boxed{\begin{aligned} x_1 - x_2 + x_3 &= 5 \\ 2x_2 &= 7 \\ x_2 - x_3 &= 6 \\ x_3 &= 1 \end{aligned}} \xrightarrow{R_1 \leftrightarrow R_3} \boxed{\begin{aligned} x_2 - x_3 &= 6 \\ 2x_2 &= 7 \\ x_1 - x_2 + x_3 &= 5 \\ x_3 &= 1 \end{aligned}}$$

Augmented matrix viewpoint

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 2 & 0 & 7 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 0 & 1 & -1 & 6 \\ 0 & 2 & 0 & 7 \\ 1 & -1 & 1 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Ex: (Adding a multiple of one row/equation to a different row/equation)

equation viewpoint

$$\boxed{\begin{aligned} x - y + z &= 3 \\ 2x + y + z &= 1 \\ y + 2z &= 10 \end{aligned}} \xrightarrow{-2R_1 + R_2 \to R_2} \boxed{\begin{aligned} x - y + z &= 3 \\ 3y - z &= -5 \\ y + 2z &= 10 \end{aligned}}$$

$$\begin{aligned} -2(x - y + z &= 3) && \leftarrow -2R_1 \\ +\ \ 2x + y + z &= 1) && \leftarrow R_2 \\ \hline 0x + 3y - z &= -5 && \leftarrow \text{new } R_2 \end{aligned}$$

Augmented matrix viewpoint

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 10 \end{array}\right) \xrightarrow{-2R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 3 & -1 & -5 \\ 0 & 1 & 2 & 10 \end{array}\right)$$

$$\begin{array}{rcl} & (-2 \quad 2 \quad -2 \mid -6) & \leftarrow -2R_1 \\ + & (2 \quad 1 \quad 1 \mid 1) & \leftarrow R_2 \\ \hline & (0 \quad 3 \quad -1 \mid -5) & \leftarrow \text{new } R_2 \end{array}$$

Theorem: Applying an elementary row operation to a system of linear equations does not change the solution space of the system

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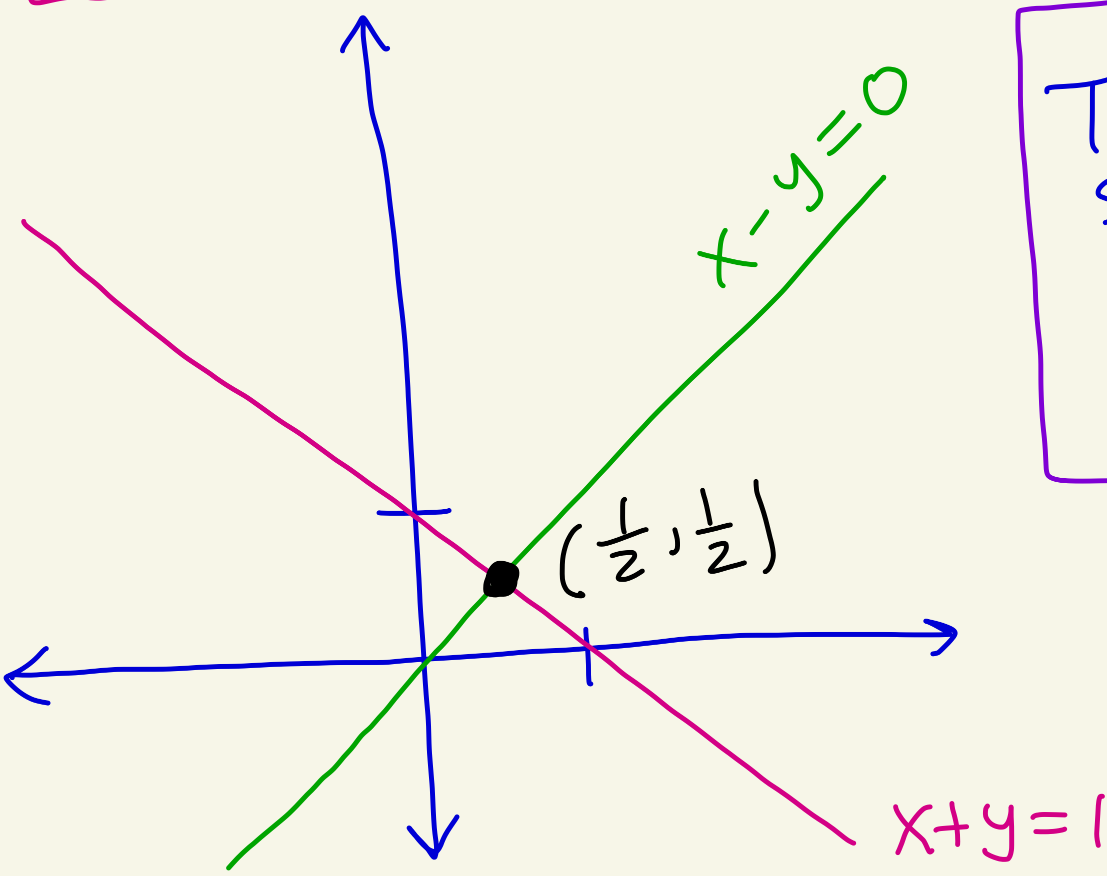
Ex:

system

$$\boxed{\begin{aligned} x+y&=1 \\ x-y&=0 \end{aligned}} \quad (*)$$

augmented matrix

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 0 \end{array}\right)$$



The solution space for $(*)$ is $\{(\frac{1}{2},\frac{1}{2})\}$

Let's apply an elementary row operation to the system (*).

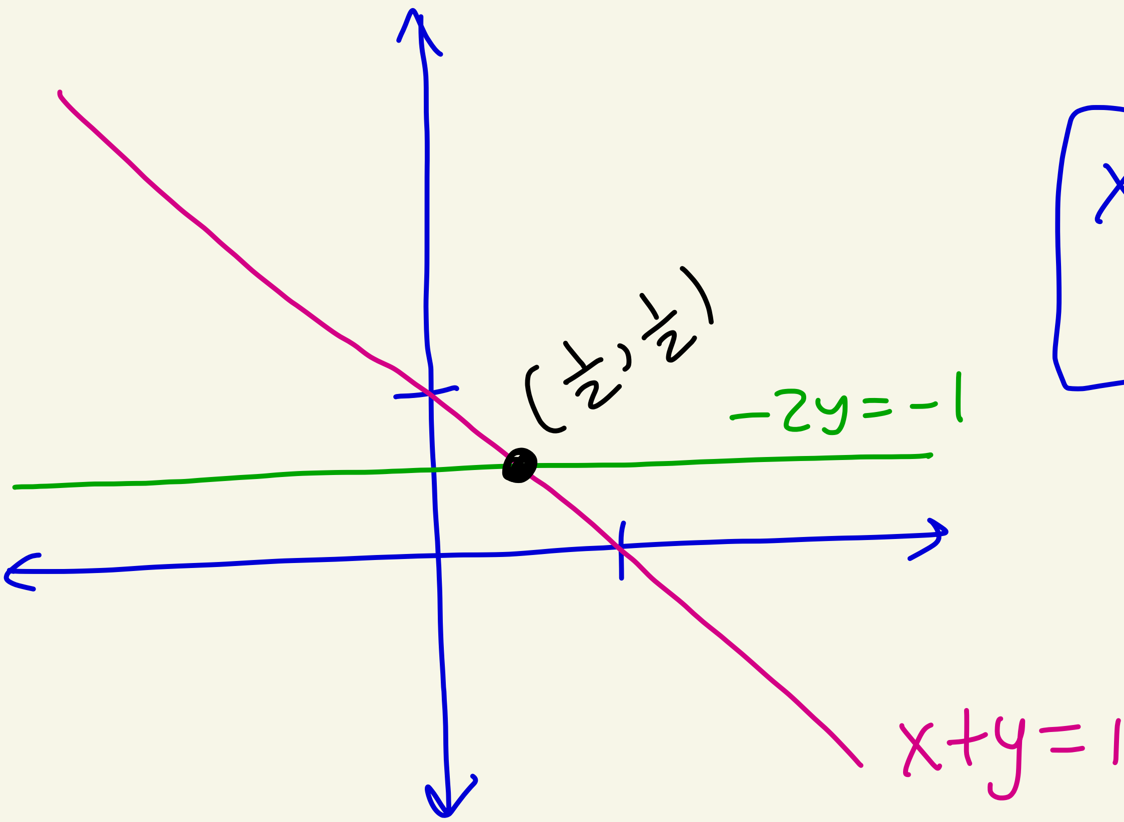
$$\underbrace{\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 0 \end{array}\right)}_{(*)} \xrightarrow{-R_1+R_2 \to R_2} \underbrace{\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & -1 \end{array}\right)}_{(**)}$$

$$\begin{array}{rl} & \left(\begin{array}{cc|c} -1 & -1 & -1 \end{array}\right) \leftarrow -R_1 \\ + & \left(\begin{array}{cc|c} 1 & -1 & 0 \end{array}\right) \leftarrow R_2 \\ \hline & \left(\begin{array}{cc|c} 0 & -2 & -1 \end{array}\right) \leftarrow \text{new } R_2 \end{array}$$

(**) corresponds to

$$\boxed{\begin{aligned} x + y &= 1 \\ -2y &= -1 \end{aligned}}$$

Drawing (**) we have



(**)

$$\boxed{\begin{aligned} x+y&=1 \\ -2y&=-1 \end{aligned}}$$

So the solution space for (**) is also $\{(\frac{1}{2}, \frac{1}{2})\}$.

We see that the elementary row operation didn't change the solutions.

Def: If a row of a matrix does not consist entirely of zeros then the leading entry in that row is the first non-zero entry when scanning from left to right

Ex:

$$A = \begin{pmatrix} 5 & 0 & 2 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & -3 & 1 & 10 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \leftarrow \text{row } 3 \\ \leftarrow \text{row } 4 \end{array}$$

leading entry in row 1 is 5
leading entry in row 2 is -1
leading entry in row 3 is -3

There is no leading entry in row 4 since row 4 consists entirely of zeros.

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Def: A matrix is in <u>row echelon form</u> if the following three conditions are true:

① If there are any rows that consist entirely of zeros, then those rows are grouped together at the bottom of the matrix.

② In any two consecutive rows that do not consist entirely of zeros, then the leading entry in the lower row occurs farther to the right than the leading entry in the higher row.

③ If a row does not consist entirely of zeros, then the leading entry of that row is $\underline{1}$.

Ex:

$$\begin{pmatrix} 1 & 5 & 10 & 7 \\ 0 & -2 & 1 & \frac{1}{2} \end{pmatrix}$$

leading entries are circled

1 ✓

2 ✓

3 X

leading entry in row 2 is not 1

This matrix <u>is not</u> in row echelon form.

$$\begin{pmatrix} 1 & 5 & 10 & 7 \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

1 ✓

2 ✓

3 ✓

This matrix <u>is</u> in row echelon form

Ex:

$$\begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

row of zeros ✓

1 ✓

2 ✗

3 ✓

$\boxed{\text{leading entry in row 3 is not to the right of the leading entry in row 2.}}$

Matrix is <u>not</u> in row echelon form.

Ex:

$$\begin{pmatrix} 1 & 2 & 5 & 7 & 6 \\ 1 & 0 & 3 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 7 & 3 \end{pmatrix}$$

1 ✗

2 ✗

3 ✓

not in row echelon form

Ex:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{leading entries circled} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{rows of zeros} \end{matrix}$$

1 ✓
2 ✓
3 ✓

This matrix is in row echelon form

<u>Def:</u> Suppose you have an augmented matrix for a system of linear equations.
Suppose you use elementary row operations to put the left side of the matrix into row echelon form.

$$\left(\begin{array}{c|c} \#\text{'s} & \begin{matrix}\#\\ \#\\ \vdots\\ \#\end{matrix}\end{array}\right)$$

left side

The variable corresponding to the leading entry of a row is called a <u>leading variable</u> (or <u>pivot variable</u>).

Any variable that doesn't occur as a leading variable is called a <u>free variable</u>

Ex:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

left side is in row echelon form

leading variables are circled

Suppose this corresponds to

$$\boxed{\begin{aligned} x \quad\quad + z &= 3 \\ y + z &= 2 \\ 0 &= 0 \end{aligned}}$$

$\leftarrow 0x+0y+0z=0$

leading variables are x and y.

Free variable is z

[Free variables are the variables that aren't leading variables]

Method to solve a system of linear equations (called Gaussian elimination)

1. Use elementary row operations to put the left side of the augmented matrix for the system into row echelon form.

2. **case (a):** If one of the equations corresponding to the reduced augmented matrix is $0 = c$ where $c \neq 0$, then the system has no solutions and we say that the system is inconsistent.

   case (b): If case (a) doesn't occur we use back-substitution to find all the solutions to the system.

To do this:

(i) Write down the equations corresponding to the reduced augmented matrix.

(ii) Solve the equations for the leading variables.

(iii) Assign each free variable a new name as these variables can take on any value.

(iv) Beginning with the bottom/last equation and working upward, successively substitute each equation into the equation above it.

Ex: Solve

$$\boxed{\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}}$$

we want a 1 here

make these into zeros

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

$$\begin{array}{rcccc|cl} & (-2 & -2 & -4 & -18) & \leftarrow -2R_1 \\ + & (2 & 4 & -3 & 1) & \leftarrow R_2 \\ \hline & (0 & 2 & -7 & -17) & \leftarrow \text{new } R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

$$\xrightarrow{-3R_1+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right)$$

put a 1 here

$$\begin{array}{rcccc|cl} & (-3 & -3 & -6 & -27) & \leftarrow -3R_1 \\ + & (3 & 6 & -5 & 0) & \leftarrow R_3 \\ \hline & (0 & 3 & -11 & -27) & \leftarrow \text{new } R_3 \end{array}$$

$$\xrightarrow{\frac{1}{2}R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array}\right)$$

make this zero

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array}\right) \xrightarrow{-3R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right)$$

turn into 1

$$\begin{array}{rl} (0 \quad -3 \quad \frac{21}{2} \mid \frac{51}{2}) & \leftarrow -3R_2 \\ +\,(0 \quad 3 \quad -11 \mid -27) & \leftarrow R_3 \\ \hline (0 \quad 0 \quad -\frac{1}{2} \mid -\frac{3}{2}) & \leftarrow \text{new } R_3 \end{array}$$

$$\xrightarrow{-2R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right)$$

left side is in row echelon form

Now turn the reduced matrix back into equations.

$$\boxed{\begin{aligned} x + y + 2z &= 9 \quad (1)\\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \quad (2)\\ z &= 3 \quad (3)\end{aligned}}$$

leading variables are x, y, z

There are no free variables

Solve in terms of leading variables.

$$\boxed{\begin{aligned} x &= 9 - y - 2z \quad (1)\\ y &= -\tfrac{17}{2} + \tfrac{7}{2}z \quad (2)\\ z &= 3 \quad (3)\end{aligned}}$$

Now we back-substitute starting at the last equation and going upwards.

(3) gives $\boxed{z=3}$

(2) gives

sub in $z=3$

$$y = -\frac{17}{2} + \frac{7}{2}z = -\frac{17}{2} + \frac{7}{2}(3)$$
$$= 2$$

So, $\boxed{y=2}$

sub in $z=3$ and $y=2$

(1) gives

$$x = 9 - y - 2z$$
$$= 9 - 2 - 2(3) = 1$$

So $\boxed{x=1}$

Thus, the only solution to the system is

$$\boxed{\begin{array}{l} x=1 \\ y=2 \\ z=3 \end{array}} \quad \text{or} \quad \boxed{(x,y,z) = (1,2,3)}$$

Let's check the answer to make sure it works

$\boxed{\text{Original System}}$

$$\left.\begin{aligned} x+y+2z&=9\\ 2x+4y-3z&=1\\ 3x+6y-5z&=0 \end{aligned}\right\} \begin{aligned} 1+2+2(3)&=9 \ \checkmark\\ 2(1)+4(2)-3(3)&=1 \ \checkmark\\ 3(1)+6(2)-5(3)&=0 \ \checkmark \end{aligned}$$

$\boxed{x=1,\ y=2,\ z=3 \text{ works}}$

This is the only solution to the system. There is no other solution (*)

Ex: Solve

$$\begin{aligned} -2b + 3c &= 1 \\ 3a + 6b - 3c &= -2 \\ 6a + 6b + 3c &= 5 \end{aligned}$$

want a 1 here

$$\left(\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right)$$

put zeros here

$$\xrightarrow{\frac{1}{3}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ \boxed{0} & -2 & 3 & 1 \\ \boxed{6} & 6 & 3 & 5 \end{array}\right)$$

$$\xrightarrow{-6R_1 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & \boxed{-2} & 3 & 1 \\ 0 & -6 & 9 & 9 \end{array}\right)$$

put a 1 here

$$\xrightarrow{-\frac{1}{2}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & \boxed{-6} & 9 & 9 \end{array}\right)$$

make this a zero

$$\xrightarrow{6R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

this left side is in row echelon form

Now we turn it back into equations.

We get

$$\boxed{\begin{aligned} a + 2b - c &= -2/3 \\ b - 3/2\, c &= -1/2 \\ 0 &= 6 \end{aligned}}$$

Since we have $0 = 6$ in the last equation this tells us that the original system is <u>inconsistent</u> that is there are <u>no solutions</u> to the system.

Ex: Solve

$$\boxed{\begin{aligned} 5x_1 - 2x_2 + 6x_3 &= 0 \\ -2x_1 + x_2 + 3x_3 &= 1 \end{aligned}}$$

put a 1 here

$$\left(\begin{array}{ccc|c} \boxed{5} & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{array}\right)$$

$$\xrightarrow{2R_2 + R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 12 & 2 \\ \boxed{-2} & 1 & 3 & 1 \end{array}\right)$$

make this a zero

Could have instead done $\frac{1}{5}R_1 \to R_1$

$$\xrightarrow{2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{array}\right)$$

this left side is in row echelon form

Turn it back into equations.

$$\boxed{\begin{aligned} x_1 \quad + 12x_3 &= 2 \quad (1) \\ x_2 + 27x_3 &= 5 \quad (2) \end{aligned}}$$

leading variables are x_1, x_2.

free variable is x_3

Solve in terms of leading variables.

$$\boxed{\begin{aligned} x_1 &= 2 - 12x_3 \quad (1) \\ x_2 &= 5 - 27x_3 \quad (2) \end{aligned}}$$

Give the free variables a new name.

Let $\boxed{x_3 = t}$

Now backsubstitute.

(2) gives $x_2 = 5 - 27x_3$

$x_3 = t$

$$\boxed{x_2 = 5 - 27t}$$

(1) gives $x_1 = 2 - 12x_3$

$$\boxed{x_1 = 2 - 12t}$$

Answer

$x_1 = 2 - 12t$
$x_2 = 5 - 27t$
$x_3 = t$

where t can be any real number

Infinitely many solutions, for example

$t = 1$

$x_1 = 2 - 12 = -10$
$x_2 = 5 - 27 = -22$
$x_3 = 1$

$t = 0$

$x_1 = 2 - 0 = 2$
$x_2 = 5 - 0 = 5$
$x_3 = 0$

Ex: Solve

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 + 15x_6 &= 5 \\
2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}$$

already a 1

make these zeros

$$\left(\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
2 & 6 & -5 & -2 & 4 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{array}\right)$$

$$\xrightarrow{-2R_1 + R_2 \to R_2}
\left(\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & -1 & -2 & 0 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{array}\right)$$

$$\xrightarrow{-2R_1+R_4\to R_4}\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

make this a 1

$$\xrightarrow{-R_2\to R_2}\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

make these zeros

$$\underset{-4R_2+R_4\to R_4}{\xrightarrow{-5R_2+R_3\to R_3}}\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right)$$

$$\xrightarrow{R_3\leftrightarrow R_4}\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{6} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

make this a 1

$$\xrightarrow{\frac{1}{6}R_3 \to R_3} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

this left side is in row echelon form

Turning this back into equations gives:

$$\boxed{\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ x_3 + 2x_4 + 3x_6 &= 1 \\ x_6 &= 1/3 \\ 0 &= 0 \end{aligned}}$$

can drop the last equation

leading variables are x_1, x_3, x_6

free variables are x_2, x_4, x_5

Solve for the leading variables:

$$\boxed{\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \quad (1) \\ x_3 &= 1 - 2x_4 - 3x_6 \quad (2) \\ x_6 &= 1/3 \quad (3) \end{aligned}}$$

Give the free variables a new name:

$x_2 = t$

$x_4 = u$

$x_5 = w$

Back substitute:

(3) gives $\boxed{x_6 = 1/3}$

(2) gives $x_3 = 1 - 2x_4 - 3x_6$

$$= 1 - 2u - 3\left(\tfrac{1}{3}\right) = -2u$$

$$\boxed{x_3 = -2u}$$

(1) gives

$$x_1 = -3x_2 + 2x_3 - 2x_5$$
$$= -3t + 2(-2u) - 2w$$
$$= -3t - 4u - 2w$$

$$\boxed{x_1 = -3t - 4u - 2w}$$

Solutions:

$x_1 = -3t - 4u - 2w$

$x_2 = t$

$x_3 = -2u$

$x_4 = u$

$x_5 = w$

$x_6 = 1/3$

where t, u, and w can be any real numbers

So we have an infinite number of solutions.

For example, if

$t=2$, $u=-3$, and $w=0$

then

$$x_1 = -3(2)-4(-3)-2(0)=6$$
$$x_2 = 2$$
$$x_3 = -2(-3)=6$$
$$x_4 = -3$$
$$x_5 = 0$$
$$x_6 = 1/3$$

is one of the infinite number of solutions.

Theorem: A system of linear equations has either

(i) no solutions

(ii) exactly one solution

or (iii) infinitely many solutions