Math 2550
HW 7 - Part 1
Spanning, Linear Independence, Bases

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1. (a) Consider the vector space $V = \mathbb{R}^2$ with field $F = \mathbb{R}$.
   List five vectors that lie in the span of $\langle 0, 1\rangle$, $\langle 1, 1\rangle$, $\langle -3, 2\rangle$.

   (b) Consider the vector space $V = \mathbb{R}^3$ with field $F = \mathbb{R}$.
   List five vectors that lie in $\text{span}(\{\langle 0, -2, 2\rangle, \langle 1, 3, -1\rangle\})$.

   (c) Consider the vector space $V = P_1$ with field $F = \mathbb{R}$.
   List five vectors that lie in the $\text{span}(\{2, 1 + x\})$.

   (d) Consider the vector space $V = P_2$ with field $F = \mathbb{R}$. List five vectors that lie in the span of $-1 - 2x$, x^2, $1 + x + x^2$.

2. Consider the vector space $V = \mathbb{R}^3$ with field $F = \mathbb{R}$. Let $\vec{u} = \langle 0, -2, 2\rangle$ and $\vec{v} = \langle 1, 3, -1\rangle$.

   Which of the following vectors lie in the span of $\vec{u}$ and $\vec{v}$? If a vector does lie in the span then express it as a linear combination of $\vec{u}$ and $\vec{v}$.

   (a) $\langle 2, 2, 2\rangle$
   (b) $\langle 3, 1, 5\rangle$
   (c) $\langle 0, 4, 5\rangle$
   (d) $\langle 0, 0, 0\rangle$

3. Consider the vector space $V = P_3$ with field $F = \mathbb{R}$. Let $\vec{p_1} = 2+x+4x^2$, $\vec{p_2} = 1 - x + 3x^2$, and $\vec{p_3} = 1 + x^3$.

   Which of the following vectors lie in $\text{span}(\{\vec{p_1}, \vec{p_2}, \vec{p_3}\})$? If a vector does lie in the span then express it as a linear combination of $\vec{p_1}$, $\vec{p_2}$, $\vec{p_3}$.

   (a) $3 + 2x + x^2 + 2x^3$
   (b) $1 + x$
   (c) 0
   (d) $4 - x + 10x^2$

4. Are the following vectors linearly independent or linearly dependent in the respective vector spaces?

   (a) $\vec{u_1} = \langle 1, -1 \rangle$, $\vec{u_2} = \langle 2, 1 \rangle$ in $V = \mathbb{R}^2$ with $F = \mathbb{R}$.

   (b) $\vec{u_1} = \langle 3, -1 \rangle$, $\vec{u_2} = \langle 4, 5 \rangle$, $\vec{u_3} = \langle -4, 7 \rangle$ in $V = \mathbb{R}^2$ with $F = \mathbb{R}$.

   (c) $\vec{v_1} = \langle -3, 0, 4 \rangle$, $\vec{v_2} = \langle 5, -1, 2 \rangle$, $\vec{v_3} = \langle 1, 1, 3 \rangle$ in $V = \mathbb{R}^3$ with $F = \mathbb{R}$.

   (d) $\vec{p_1} = 3 - 2x + x^2$, $\vec{p_2} = 1 + x + x^2$, $\vec{p_3} = 6 - 4x + 2x^2$ in $V = P_2$ with $F = \mathbb{R}$.

   (e) $\vec{p_1} = 1$, $\vec{p_2} = 1 + x$, $\vec{p_3} = 1 + x + x^2$ in $V = P_2$ with $F = \mathbb{R}$.

5. Determine if the following vectors are linearly independent or linearly dependent in $\mathbb{R}^3$.

   (a) $\vec{v_1} = \langle 2, 2, 2 \rangle$, $\vec{v_2} = \langle 4, 1, 2 \rangle$, $\vec{v_3} = \langle 0, 1, 1 \rangle$

   (b) $\vec{v_1} = \langle 2, -1, 3 \rangle$, $\vec{v_2} = \langle 4, 1, 2 \rangle$, $\vec{v_3} = \langle 8, -1, 8 \rangle$

6. Determine which of the sets of vectors from problem (5) above is a basis for $\mathbb{R}^3$.

7. Determine which of the following sets is a basis for $V = \mathbb{R}^3$ over the field $F = \mathbb{R}$.

   (a) $\vec{v_1} = \langle 4, -1, 2 \rangle$, $\vec{v_2} = \langle -4, 10, 2 \rangle$

   (b) $\vec{v_1} = \langle -3, 0, 4 \rangle$, $\vec{v_2} = \langle 5, -1, 2 \rangle$, $\vec{v_3} = \langle 1, 1, 3 \rangle$

   (c) $\vec{v_1} = \langle -2, 0, 1 \rangle$, $\vec{v_2} = \langle 3, 2, 5 \rangle$, $\vec{v_3} = \langle 6, -1, 1 \rangle$, $\vec{v_4} = \langle 7, 0, -2 \rangle$

8. Determine which of the following sets is a basis for $V = P_2$ over the field $F = \mathbb{R}$.

   (a) $\vec{p_1} = 1$, $\vec{p_2} = 1 + x$, $\vec{p_3} = 1 + x + x^2$

   (b) $\vec{p_1} = 6 - x^2$, $\vec{p_2} = 1 + x + 4x^2$, $\vec{p_3} = 8 + 2x + 7x^2$

9. (a) Show that the two vectors $\{\langle 1, 4 \rangle, \langle 3, -2 \rangle\}$ form a basis for $\mathbb{R}^2$.

   (b) Find the coordinates of $\langle -7, 14 \rangle$ with respect to the ordered basis $\beta = [\langle 1, 4 \rangle, \langle 3, -2 \rangle]$.

   (c) Find the coordinates of $\langle 3, -12 \rangle$ with respect to the ordered basis $\beta = [\langle 1, 4 \rangle, \langle 3, -2 \rangle]$.

10. (a) Show that the vectors $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$ form a basis for $M_{2,2}$.

(b) Find the coordinates of $\begin{pmatrix} 1 & -2 \\ 0 & -3 \end{pmatrix}$ with respect to the ordered basis $\beta = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right]$

(c) Find the coordinates of $\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}$ with respect to the ordered basis $\beta = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right]$

11. In problem 8a you showed that $1, 1 + x, 1 + x + x^2$ is a basis for P_2.

(a) Find the coordinates of $1 - x + 2x^2$ with respect to the ordered basis $\beta = [1, 1 + x, 1 + x + x^2]$.

(b) Find the coordinates of x with respect to the ordered basis $\beta = [1, 1 + x, 1 + x + x^2]$.

12. Show that the vector space $V = M_{2,2}$ of all 2×2 matrices over the field $F = \mathbb{R}$ has dimension 4.

13. Show that the vector space $V = P_n$ over the field $F = \mathbb{R}$ has dimension $n + 1$.