<u>Def:</u> The $n \times n$ <u>identity matrix</u>, denoted by I_n [or just I when we don't want to or need to say the size], is the $n \times n$ matrix with 1's along the main diagonal and 0's everywhere else.

Ex:

$$I_1 = \begin{pmatrix} 1 \end{pmatrix}$$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and so on.

<u>Ex:</u> Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \leftarrow \boxed{2\times 2}$

Consider

$$I = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We have that

$$A\,I = \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{2\times 2}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{2\times 2}$$

equal ✓

answer is 2×2

$$= \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 3 & 4 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} (1)(1)+(2)(0) & (1)(0)+(2)(1) \\ (3)(1)+(4)(0) & (3)(0)+(4)(1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = A$$

So, $A I_2 = A$.

You can also calculate $\underbrace{I_2}_{2\times 2} \underbrace{A}_{2\times 2}$ which is defined $\underbrace{\quad}_{\checkmark}$

and you will get $I_2 A = A$.

Ex:

Let $A = \begin{pmatrix} 1 & 3 & \pi \\ -1 & 2 & -2 \end{pmatrix} \leftarrow \boxed{2 \times 3}$

Note that

$$\underbrace{I_2}_{2 \times 2} \underbrace{A}_{2 \times 3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & \pi \\ -1 & 2 & -2 \end{pmatrix}$$

$\checkmark$

answer $= 2 \times 3$

$$= \begin{pmatrix} (1\ 0) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} & (1\ 0) \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} & (1\ 0) \cdot \begin{pmatrix} \pi \\ -2 \end{pmatrix} \\ (0\ 1) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} & (0\ 1) \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} & (0\ 1) \cdot \begin{pmatrix} \pi \\ -2 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & \pi \\ -1 & 2 & -2 \end{pmatrix} = A$$

Note that $\underbrace{A}_{2\times 3}\,\underbrace{I_2}_{2\times 2}$ is not defined.

$\neq$

But if you calculate

$$\underbrace{A}_{2\times 3}\,\underbrace{I_3}_{3\times 3} = \begin{pmatrix} 1 & 3 & \pi \\ -1 & 2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$=$

$$= \begin{pmatrix} 1 & 3 & \pi \\ -1 & 2 & -2 \end{pmatrix} = A$$

you fill in this part

So, $AI_3 = A$.

Theorem:

Let A, B, C be matrices and let α, β be real numbers.

α = alpha
β = beta

Then the following are true where we will assume that the sizes of the matrices are such that the operations are defined:

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $A(BC) = (AB)C$
4. $A(B + C) = AB + AC$
5. $(B + C)A = BA + CA$
6. $A(B - C) = AB - AC$

7. $(B-C)A = BA - CA$
8. $\alpha(B+C) = \alpha B + \alpha C$
9. $\alpha(B-C) = \alpha B - \alpha C$
10. $(\alpha+\beta)A = \alpha A + \beta A$
11. $(\alpha-\beta)A = \alpha A - \beta A$
12. $\alpha(\beta A) = (\alpha\beta)A$
13. $\alpha(AB) = (\alpha A)B = A(\alpha B)$
14. $(A^T)^T = A$
15. $(A+B)^T = A^T + B^T$
16. $(A-B)^T = A^T - B^T$
17. $(\alpha A)^T = \alpha A^T$
18. $(AB)^T = B^T A^T$

note the reversal of the order

19 If A is $m \times n$, then

$$A I_n = A.$$

20 If A is $m \times n$, then

$$I_m A = A$$

21 If A is $m \times n$, then

$$A - A = O_{m \times n}$$

22 If A is $m \times n$, then

$$A + O_{m \times n} = O_{m \times n} + A = A$$

Lets prove part 5 of the previous theorem for 2×2 matrices.
[This is HW 2-Part 2 #1(a)]

Suppose that A, B, C are 2×2 matrices.] Given

Prove that
$$(B+C)A = BA + CA$$] what you need to show

<u>proof:</u>

Let A, B, C be 2×2 matrices.] declare what the objects are

Then,
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \quad C = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$$

where $a, b, c, d, e, f, g, h, i, j, k, l$ are real numbers.

Then,

$$(B+C)A = \underbrace{\left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} i & j \\ k & l \end{pmatrix}\right]}_{\text{do first because of parentheses}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \underbrace{\underbrace{\begin{pmatrix} e+i & f+j \\ g+k & h+l \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{2\times 2}}_{\text{answer is } 2\times 2}$$

$$= \begin{pmatrix} (e+i \quad f+j)\cdot\begin{pmatrix} a \\ c \end{pmatrix} & (e+i \quad f+j)\cdot\begin{pmatrix} b \\ d \end{pmatrix} \\ (g+k \quad h+l)\cdot\begin{pmatrix} a \\ c \end{pmatrix} & (g+k \quad h+l)\cdot\begin{pmatrix} b \\ d \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} (e+i)a+(f+j)c & (e+i)b+(f+j)d \\ (g+k)a+(h+l)c & (g+k)b+(h+l)d \end{pmatrix}$$

$$= \begin{pmatrix} ea+ia+fc+jc & eb+ib+fd+jd \\ ga+ka+hc+lc & gb+kb+hd+ld \end{pmatrix} \quad (*)$$

We also have that

$$BA+CA = \underbrace{\underbrace{\begin{pmatrix} e & f \\ g & h \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{2\times 2}}_{\text{answer is } 2\times 2} + \underbrace{\underbrace{\begin{pmatrix} i & j \\ k & l \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{2\times 2}}_{\text{answer is } 2\times 2}$$

$$= \begin{pmatrix} (e\ f)\cdot\begin{pmatrix} a \\ c \end{pmatrix} & (e\ f)\cdot\begin{pmatrix} b \\ d \end{pmatrix} \\ (g\ h)\cdot\begin{pmatrix} a \\ c \end{pmatrix} & (g\ h)\cdot\begin{pmatrix} b \\ d \end{pmatrix} \end{pmatrix}$$

$$+ \begin{pmatrix} (i\ j)\cdot\begin{pmatrix} a \\ c \end{pmatrix} & (i\ j)\cdot\begin{pmatrix} b \\ d \end{pmatrix} \\ (k\ l)\cdot\begin{pmatrix} a \\ c \end{pmatrix} & (k\ l)\cdot\begin{pmatrix} b \\ d \end{pmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{pmatrix} + \begin{pmatrix} ia+jc & ib+jd \\ ka+lc & kb+ld \end{pmatrix}$$

$$= \begin{pmatrix} ea+fc+ia+jc & eb+fd+ib+jd \\ ga+hc+ka+lc & gb+hd+kb+ld \end{pmatrix} \quad (**)$$

We can see that $(*)$ equals $(**)$.

Thus,

$$(B+C)A = BA + CA.$$

$\blacksquare$

(end of proof symbol)

other ways: QED, $\square$

Let's prove part 15 for 3×2 matrices.
[This is HW 2 - Part 2 #1(f)]

Let A and B be 3×2 matrices.] Given

Prove that
$$(A+B)^T = A^T + B^T$$] Need to show

<u>Proof:</u>

Let A and B be 3×2 matrices.] declare the objects

Then,
$$A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \text{ and } B = \begin{pmatrix} g & h \\ i & j \\ k & l \end{pmatrix}$$

where $a, b, c, d, e, f, g, h, i, j, k, l$ are real numbers.

We have that

$$(A+B)^T = \left(\underbrace{\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} + \begin{pmatrix} g & h \\ i & j \\ k & l \end{pmatrix}}_{\text{add first because of parentheses}} \right)^T$$

$$= \begin{pmatrix} a+g & b+h \\ c+i & d+j \\ e+k & f+l \end{pmatrix}^T = \begin{pmatrix} a+g & c+i & e+k \\ b+h & d+j & f+l \end{pmatrix}$$

$$= \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} + \begin{pmatrix} g & i & k \\ h & j & l \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}^T + \begin{pmatrix} g & h \\ i & j \\ k & l \end{pmatrix}^T$$

$$= A^T + B^T \qquad \blacksquare$$