Math 2150-02
9/8/25

Ex: Solve $\frac{dy}{dx} = \frac{-x}{y}$

subject to $y(4) = 3$

We have

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

Let's find C before we solve for y. We will

Use $y(4)=3$ condition.

Plug in $x=4$ and $y=3$

to get

$$\frac{3^2}{2} = -\left(\frac{4^2}{2}\right) + C$$

$$\frac{9}{2} = -8 + C$$

$$\frac{25}{2} = C$$

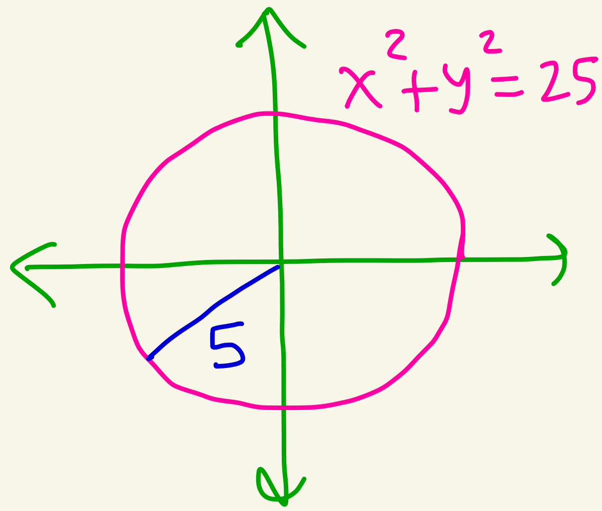
Thus,

$$\frac{y^2}{2} = -\frac{x^2}{2} + \frac{25}{2}$$

So,

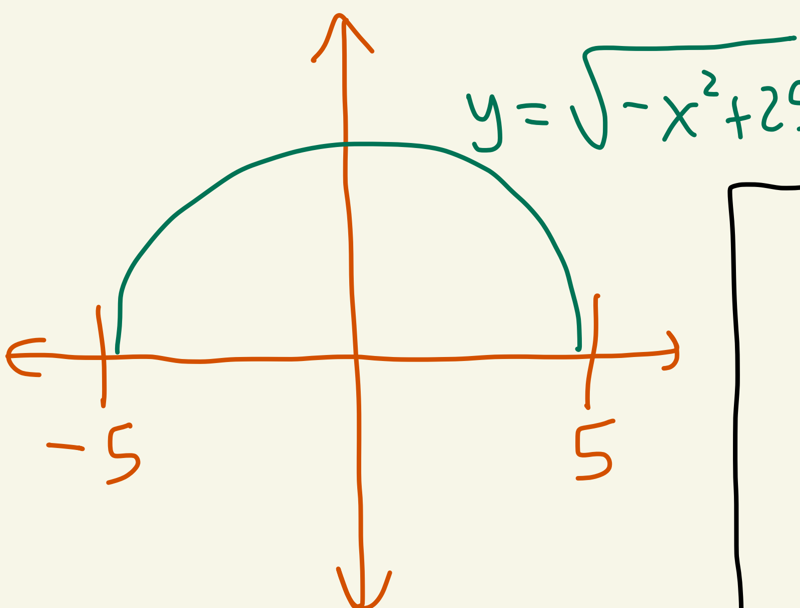
$$y^2 = -x^2 + 25$$

$$x^2 + y^2 = 25$$

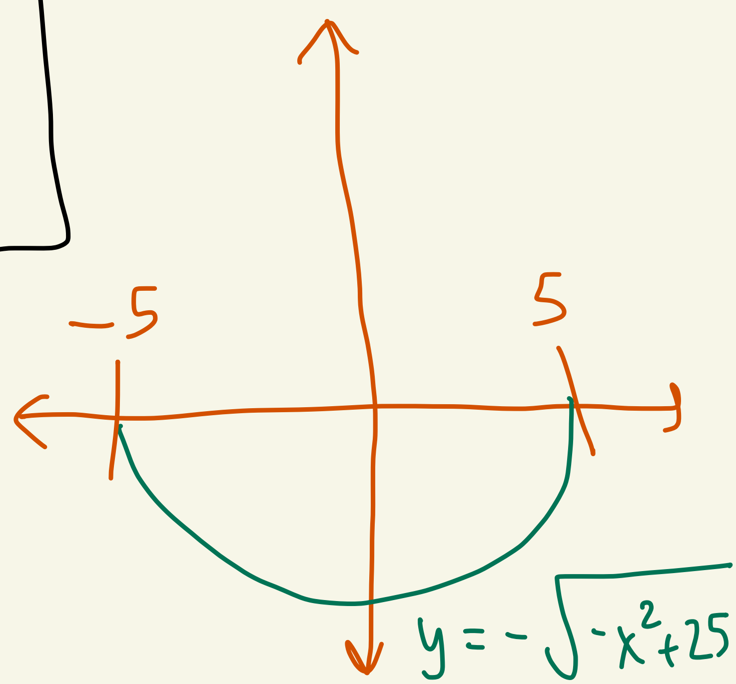


Solve for y to get:

$$y = \pm\sqrt{-x^2+25}$$



$$y(4) = 3$$



$$y(4) = -3$$

The answer is

$$y = \sqrt{-x^2 + 25}$$

because we need $y(4) = 3$.

This solution is valid for

$$-5 \leq x \leq 5$$

Topic 5 - First order Exact Equations

Suppose you have a first order equation of the form

$$\underbrace{M(x,y)}_{} + \underbrace{N(x,y)}_{} \cdot y' = 0$$

these have #, x, y
but no y'

<u>Ex:</u> $$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)} y' = 0$$

Suppose there exists a function $f(x,y)$ where

$$\frac{\partial f}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x,y)$$

Ex: $\underbrace{2xy}_{M} + \underbrace{(x^2-1)}_{N}y' = 0$

Let $f(x,y) = x^2y - y$

Then,

$$\frac{\partial f}{\partial x} = 2xy - 0 = 2xy = M(x,y)$$

$$\frac{\partial f}{\partial y} = x^2 - 1 = N(x,y)$$

Continuing on...

Suppose $\frac{\partial f}{\partial x} = M(x,y)$, $\frac{\partial f}{\partial y} = N(x,y)$.

Then,

$$M(x,y) + N(x,y) \cdot y' = 0$$

becomes

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$$

Recall from Calc III

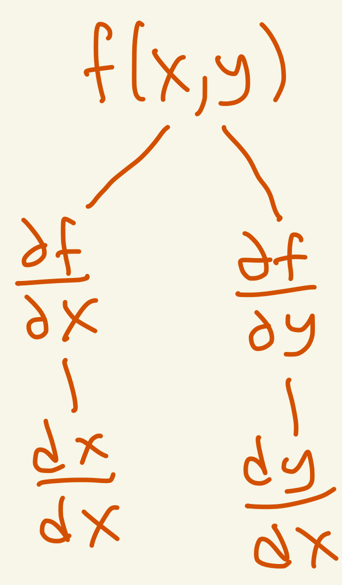
$f(x,y)$ is a function of x and y

$y = y(x)$ is a function of x

x is the variable.

Chain rule says:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$



$$= \frac{\partial f}{\partial x} \cdot 1 + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

So, $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$

becomes $\frac{df}{dx} = 0$.

A solution to $\frac{df}{dx} = 0$ can be gotten from the family of curves $f(x,y) = c$ where c is any constant.

Summary: If

$\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$

then the equation

$$f(x,y) = c$$

where c is any constant, will give an implicit solution to $M(x,y) + N(x,y) \cdot y' = 0$.

If such an f exists then we call

$$M(x,y) + N(x,y) \cdot y' = 0$$

an exact equation

Ex: $\underbrace{2xy}_{M} + \underbrace{(x^2-1)}_{N} y' = 0$

Let $f(x,y) = x^2y - y$.

We saw

$$\frac{\partial f}{\partial x} = 2xy = M(x,y)$$

$$\frac{\partial f}{\partial y} = x^2 - 1 = N(x,y)$$

Thus,

$$\boxed{x^2y - y = C} \leftarrow \boxed{f(x,y) = c}$$

will give a solution to

$$2xy + (x^2-1)y' = 0$$

where c can be any constant.

Let's verify that this works.

Verification #1

Differentiate the solution

$$x^2 y - y = c$$

with respect to x to get:

$$2xy + x^2 \underbrace{y'}_{\frac{dy}{dx}} - \underbrace{y'}_{\frac{dy}{dx}} = 0$$

$$2xy + (x^2-1)y' = 0$$

which is the equation we wanted to solve

Verification #2:

Solve $x^2 y - y = c$ for y to get:

$$y(x^2-1) = c$$

$$y = \frac{c}{x^2-1}$$ the solution

Let's check that it works.

We have

$$y = c(x^2-1)^{-1} = \frac{c}{x^2-1}$$

$$y' = -c(x^2-1)^{-2} \cdot (2x) = \frac{-2xc}{(x^2-1)^2}$$

Plug into left side of

$2xy + (x^2-1)y' = 0$ to get:

$$2xy + (x^2-1)y'$$

$$= 2x\left(\frac{c}{x^2-1}\right) + (x^2-1)\left(\frac{-2xc}{(x^2-1)^2}\right)$$

$$= \frac{2xc}{x^2-1} - \frac{2xc}{x^2-1} = 0$$

So, $y = \dfrac{c}{x^2-1}$ solves

$$2xy + (x^2-1)y' = 0$$