Math 2150-02

9/3/25

(Topic 3 continued...)

Ex: (HW 3 – 2(b))

Find all solutions to

$$x^2 y' + x(x+2) y = e^x$$

on $I = (0, \infty)$

means: want solution when $0 < x$

Step 1: Divide by x^2 to put a 1 in front of y'.

We get:

$$y' + \frac{x(x+2)}{x^2} y = \frac{e^x}{x^2}$$

We have:

$$y' + \underbrace{\left(1+\frac{2}{x}\right)}y = \frac{e^x}{x^2}$$

Step 2: Do what we did last week.

Let $A(x) = \int \left(1+\frac{2}{x}\right)dx$

$$= x + 2\ln|x|$$

$$= x + 2\ln(x)$$

$x>0$ in this problem

We need

$$e^{A(x)} = e^{x+2\ln(x)}$$

$e^{A+B} = e^A e^B$

$$= e^x e^{2\ln(x)}$$

$A\ln(B)$

$$= e^x \underbrace{e^{\ln(x^2)}}$$

$= \ln(B^A)$

$e^{\ln(A)} = A$

$$= x^2 e^x$$

Multiply $y' + \left(1 + \frac{2}{x}\right)y = \frac{e^x}{x^2}$

by $e^{A(x)} = x^2 e^x$ to get:

$$\underbrace{x^2 e^x y' + x^2 e^x \left(1 + \frac{2}{x}\right)y}_{\text{always } \left(e^{A(x)} \cdot y\right)'} = x^2 e^x \cdot \frac{e^x}{x^2}$$

we get:

$$\left(x^2 e^x \cdot y\right)' = e^{2x}$$

$e^x e^x$
$= e^{x+x}$
$= e^{2x}$

Integrate:

$$x^2 e^x \cdot y = \underbrace{\int e^{2x}\, dx}$$

$$\int e^{2x}\, dx \underset{\uparrow}{=} \int \frac{1}{2} e^u\, du$$

$$\boxed{\begin{aligned} u &= 2x \\ du &= 2dx \\ \tfrac{1}{2} du &= dx \end{aligned}}$$

$$= \frac{1}{2} e^u + C$$

$$= \frac{1}{2} e^{2x} + C$$

We get

$$x^2 e^x y = \frac{1}{2} e^{2x} + C$$

So,

$$y = \frac{e^{2x}}{2x^2e^x} + \frac{c}{x^2e^x}$$

$$\frac{e^{2x}}{e^x} = e^{2x-x} = e^x$$

Thus,

$$y = \frac{e^x}{2x^2} + \frac{c}{x^2e^x}$$

Answer:

$$\boxed{y = \frac{e^x}{2x^2} + \frac{c}{x^2e^x}}$$

where C is any constant

Topic 4 - Separable first order ODEs

<u>Def:</u> A first order ODE is called separable if it is of the form

$$\underbrace{N(y)}_{\text{\#'s and } y\text{'s}} \cdot y' = \underbrace{M(x)}_{\text{\#'s and } x\text{'s}}$$

<u>Ex:</u>

$$\underbrace{(y^2-2)}_{N(y)}\, y' = \underbrace{\cos(3x)}_{M(x)}$$

Ex: Is $y' = \frac{x^2}{y}$ separable?

Yes, multiply by y to get:

$$\underbrace{y}_{N(y)}y' = \underbrace{x^2}_{M(x)}$$

How to solve a separable ODE

Formal notation	Differential form notation
$N(y)\cdot y' = M(x)$	$N(y)\, y' = M(x)$
$\downarrow$	$\downarrow$
$N(y(x))\cdot y'(x) = M(x)$	$N(y)\,\dfrac{dy}{dx} = M(x)$
$\downarrow$	$\downarrow$
$\int N(y(x))\, y'(x)\, dx = \int M(x)\, dx$	$N(y)\, dy = M(x)\, dx$
$\downarrow$ $\boxed{u = y(x),\ du = y'(x)\, dx}$	$\downarrow$
$\int N(u)\, du = \int M(x)\, dx$	$\int N(y)\, dy = \int M(x)\, dx$
Now integrate	Now integrate
Here $u = y$	

Ex: Solve the initial-value problem

$$\boxed{\begin{aligned} y^2 \frac{dy}{dx} &= x-5 \\ y(0) &= 1 \end{aligned}}$$

Furthermore, give an interval where the solution exists.

We have

$$y^2 \frac{dy}{dx} = x-5$$

separable

$$y^2 dy = (x-5)dx$$

$$\int y^2 dy = \int (x-5) dx$$

$$\frac{1}{3}y^3 = \frac{1}{2}x^2 - 5x + C$$

Before solving for y let's use the $y(0) = 1$ condition.

$\boxed{x}$

Plug in $x = 0$, $y = 1$ to get:

$$\frac{1}{3}(1)^3 = \frac{1}{2}0^2 - 5(0) + C$$

$$\frac{1}{3} = C$$

So,

$$\frac{1}{3}y^3 = \frac{1}{2}x^2 - 5x + \frac{1}{3}$$

Thus,

$$y = \left(\frac{3}{2}x^2 - 15x + 1\right)^{1/3}$$

This solution is valid on $I = (-\infty, \infty)$ $\boxed{-\infty < x < \infty}$