Math 2150-02
9/24/25

(Topic 7 continued...)

Ex: Solve

$$4y'' - y' = 0$$
$$y(0) = -1$$
$$y'(0) = 1$$

The characteristic poly. is

$$4r^2 - r = 0$$

$$r\underbrace{(4r-1)} = 0$$

$\boxed{r = 0}$

$\boxed{\begin{array}{c} 4r - 1 = 0 \\ r = 1/4 \end{array}}$

The general solution

to $4y'' - y' = 0$ is

$$y_h = c_1 e^{0x} + c_2 e^{\frac{1}{4}x}$$

So,

$$\boxed{y_h = c_1 + c_2 e^{x/4}}$$

Let's make it satisfy

$$\boxed{\begin{aligned} y_h(0) &= -1 \\ y_h'(0) &= 1 \end{aligned}}$$

We have

$$y_h = c_1 + c_2 e^{x/4}$$

$$y_h' = \frac{1}{4} c_2 e^{x/4}$$

We want:

$$\boxed{\begin{aligned} c_1 + c_2 e^{0/4} &= -1 \\ \tfrac{1}{4} c_2 e^{0/4} &= 1 \end{aligned}} \leftarrow \boxed{\begin{aligned} y_h(0) &= -1 \\ y_h'(0) &= 1 \end{aligned}}$$

$$\downarrow$$

$$\boxed{\begin{aligned} c_1 + c_2 &= -1 \\ \tfrac{1}{4} c_2 &= 1 \end{aligned}} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

(2) gives us $c_2 = 4$.

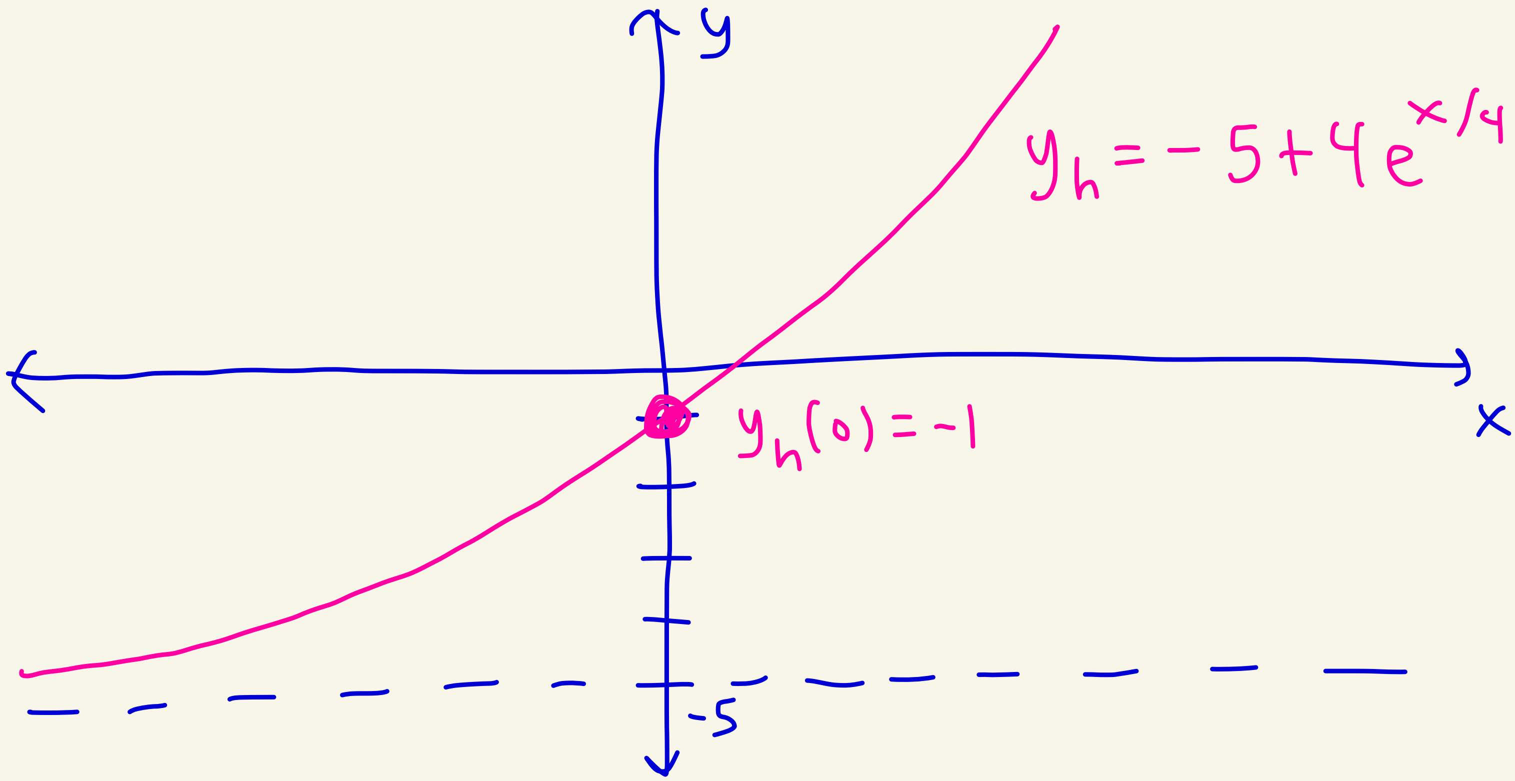
Plug into (1) to get $c_1 = -1 - c_2$
$= -1 - 4 = -5$

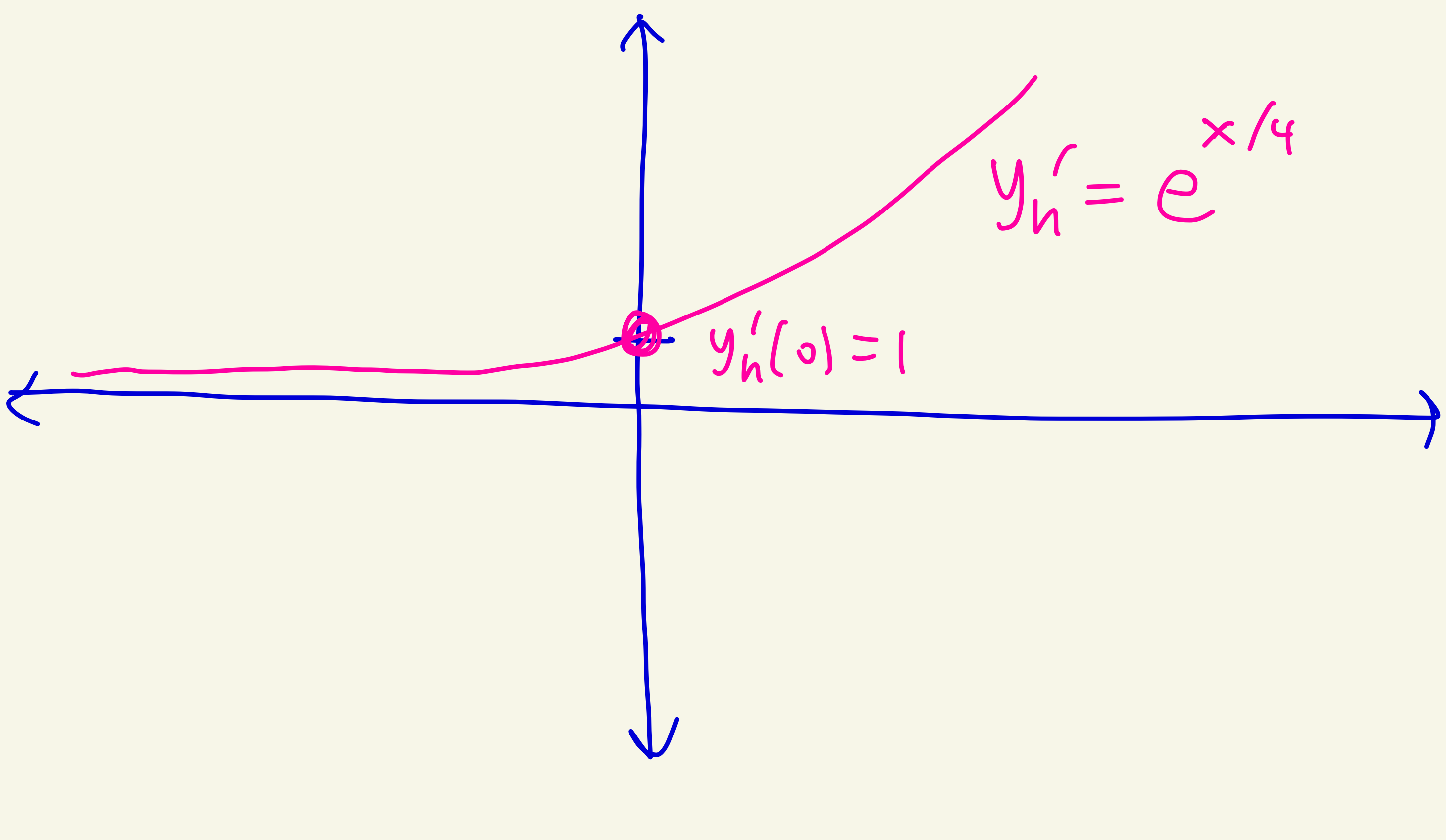
Thus, the solution to

$4y'' - y' = 0$, $y(0) = -1$, $y'(0) = 1$

is

$$y_h = -5 + 4e^{x/4}$$





Why do the formulas from last time work?

Let's look at case 1.

Suppose we are looking at

$$a_2 y'' + a_1 y' + a_0 y = 0$$

where a_2, a_1, a_0 are constants and $a_2 \neq 0$.

Let's try $y = e^{rx}$ where r is a constant.

We have

$$y = e^{rx}$$

$$y' = re^{rx}$$

$$y'' = r^2 e^{rx}$$

If we plug this into

$$a_2 y'' + a_1 y' + a_0 y = 0$$

we have

$$a_2(r^2 e^{rx}) + a_1(r e^{rx}) + a_0(e^{rx}) = 0$$

This becomes

$$e^{rx}\left[a_2 r^2 + a_1 r + a_0\right] = 0$$

never 0

this will be 0 when r is a root of the characteristic poly.

$$a_2 r^2 + a_1 r + a_0 = 0$$

So if we are in case 1 where $a_2 r^2 + a_1 r + a_0 = 0$

has two distinct real roots r_1, r_2 $(r_1 \neq r_2)$ then $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ will both be solutions to

$$a_2 y'' + a_1 y' + a_0 = 0.$$

Are y_1 and y_2 linearly independent?

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix}$$

$$= (e^{r_1 x})(r_2 e^{r_2 x}) - (r_1 e^{r_1 x})(e^{r_2 x})$$

$$= r_2 e^{r_1 x + r_2 x} - r_1 e^{r_1 x + r_2 x}$$

$$= \underbrace{(r_2 - r_1)}_{\substack{r_2 - r_1 \neq 0 \\ \text{because} \\ r_2 \neq r_1}} \underbrace{e^{r_1 x + r_2 x}}_{\substack{\text{never} \\ 0}}$$

Thus, $W(y_1, y_2)$ is never 0.

So, $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are linearly independent solutions to

$$a_2 y'' + a_1 y' + a_0 y = 0$$

for $I = (-\infty, \infty)$.

Thus, the general solution is

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Cases 2/3 are online...