Math 2150-02
9/17/25

Last time we saw that the general solution to

$$y'' - 7y' + 10y = 24e^x$$

on $I = (-\infty, \infty)$ is

$$y = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{\substack{\text{homogeneous solution } y_h \text{ to} \\ y'' - 7y' + 10y = 0}} + \underbrace{6e^x}_{\substack{\text{particular solution } y_p \text{ to} \\ y'' - 7y' + 10y = 24e^x}}$$

where c_1, c_2 are constants.

We have an infinite # of solutions, one for each choice of c_1 & c_2.

But if we turn it into an initial-value problem then we will only get one solution.

<u>Ex:</u> Solve

$$\boxed{\begin{aligned} &y'' - 7y' + 10y = 24e^{x} \\ &\left.\begin{aligned} &y(0) = 0 \\ &y'(0) = 1 \end{aligned}\right\} \end{aligned}}$$

ODE

initial conditions

From last time, the general solution to $y'' - 7y' + 10y = 24e^{x}$ is

$$y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$$

We have

$$y' = 2c_1 e^{2x} + 5c_2 e^{5x} + 6e^{x}$$

We get:

$$\boxed{\begin{aligned} y(0) &= 0 \\ y'(0) &= 1 \end{aligned}}$$

$$\boxed{\begin{aligned} c_1 e^{2(0)} + c_2 e^{5(0)} + 6e^0 &= 0 \\ 2c_1 e^{2(0)} + 5c_2 e^{5(0)} + 6e^0 &= 1 \end{aligned}}$$

$e^0 = 1$

$$\boxed{\begin{aligned} c_1 + c_2 + 6 &= 0 \\ 2c_1 + 5c_2 + 6 &= 1 \end{aligned}}$$

$$\boxed{\begin{aligned} c_1 + c_2 &= -6 \quad (1) \\ 2c_1 + 5c_2 &= -5 \quad (2) \end{aligned}}$$

(1) gives $c_1 = -6 - c_2$.

Plug into (2) to get:

$$2(-6 - c_2) + 5c_2 = -5$$

So,

$$3c_2 = 7$$

Thus, $c_2 = 7/3$

Ergo, $c_1 = -6 - c_2$

$$= -6 - 7/3$$

$$= \frac{-18}{3} - \frac{7}{3} = \frac{-25}{3}$$

Thus the solution to

$$y'' - 7y' + 10y = 24e^x$$
$$y(0) = 0, \quad y'(0) = 1$$

is

$$y = c_1 e^{2x} + c_2 e^{5x} + 6e^x$$
$$= \frac{-25}{3} e^{2x} + \frac{7}{3} e^{5x} + 6e^x$$

$\boxed{\text{Topic 7 - 2nd order linear homogeneous constant coefficient ODEs}}$

We now learn how to solve

$$a_2 y'' + a_1 y' + a_0 y = 0$$

where a_2, a_1, a_0 are constants.

<u>Ex:</u> $y'' - 7y' + 10y = 0$

$\boxed{a_2 = 1}$ $\boxed{a_1 = -7}$ $\boxed{a_0 = 10}$

Def: The characteristic equation of

$$a_2 y'' + a_1 y' + a_0 y = 0$$

is

$$a_2 r^2 + a_1 r + a_0 = 0$$

where r is a number/variable.

Ex: The characteristic equation of

$$y'' - 7y' + 10y = 0$$

is

$$r^2 - 7r + 10 = 0$$

Formula Time

Consider

$$\boxed{a_2 y'' + a_1 y' + a_0 y = 0} \quad (*)$$

where a_2, a_1, a_0 are constants and assume $a_2 \neq 0$.

There are three cases, depending on the roots of the characteristic polynomial

$$a_2 r^2 + a_1 r + a_0 = 0.$$

<u>case 1:</u> Suppose the characteristic polynomial has two distinct real roots r_1, r_2 $(r_1 \neq r_2)$

then the solution to (*) is

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

<u>Case 2:</u> Suppose the characteristic polynomial has a repeated real root r (only one root), then the solution to (*) is

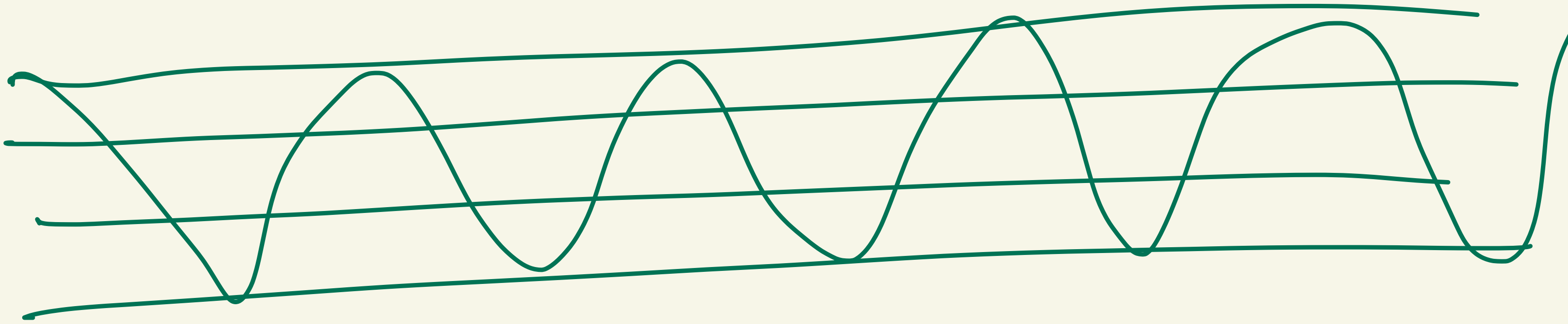
$$y_h = c_1 e^{rx} + c_2 x e^{rx}$$

<u>Case 3:</u> Suppose the characteristic polynomial has imaginary/complex roots $\alpha \pm i\beta$

α = alpha
β = beta
$i = \sqrt{-1}$

then the solution to (*) is

$$y_h = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$



Ex: Solve $y'' - 7y' + 10y = 0$.

The characteristic polynomial is

$$r^2 - 7r + 10 = 0$$

The roots are

$$r = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(10)}}{2(1)}$$

$$= \frac{7 \pm \sqrt{9}}{2} = \frac{7 \pm 3}{2}$$

$$= \frac{7+3}{2}, \frac{7-3}{2} = \boxed{5, 2}$$

case 1
two real
roots

Answer:

$$y_h = c_1 e^{2x} + c_2 e^{5x}$$

Ex: Solve $y'' - 4y' + 4y = 0$.

The characteristic equation is

$$r^2 - 4r + 4 = 0$$

The roots are:

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{0}}{2} = \frac{4}{2} = \boxed{2}$$

case 2
one real
root

The solution is

$$y_h = c_1 e^{2x} + c_2 x e^{2x}$$

Ex: Solve $y'' - 4y' + 13y = 0$

The characteristic equation is

$$r^2 - 4r + 13 = 0$$

The roots are

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16-52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm \sqrt{36}\sqrt{-1}}{2}$$

$i = \sqrt{-1}$

$i^2 = -1$

$$= \frac{4 \pm 6i}{2} = \frac{4}{2} \pm \frac{6}{2} i$$

$$= \underbrace{2 \pm 3i}_{\alpha \pm \beta i}$$

$\alpha = 2, \ \beta = 3$

The solution is

$$y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

$$= c_1 e^{2x} \cos(3x) + c_2 e^{2x} \sin(3x)$$