Math 2150-02
9/17/25

Ex: Let's solve

$$y'' - 7y' + 10y = 24e^{x}$$

on $I = (-\infty, \infty)$ ⟵ $\boxed{-\infty < x < \infty}$

Step 1: Find two linearly independent solutions to

$$y'' - 7y' + 10y = 0$$

homogeneous equation

Let

$$f_1(x) = e^{2x}$$

$$f_2(x) = e^{5x}$$

in topic 7 we will see how to find these

Last time we showed that f_1 and f_2 are linearly independent. Let's show they solve $y'' - 7y' + 10y = 0$.

For f_1 we get:

$f_1 = e^{2x}$, $f_1' = 2e^{2x}$, $f_1'' = 4e^{2x}$

So,

$$\begin{aligned} f_1'' - 7f_1' + 10f_1 &= 4e^{2x} - 7(2e^{2x}) + 10(e^{2x}) \\ &= 4e^{2x} - 14e^{2x} + 10e^{2x} \\ &= 0 \end{aligned}$$

So, f_1 solves the ODE.

Let's try f_2.

We get

$f_2 = e^{5x}$, $f_2' = 5e^{5x}$, $f_2'' = 25e^{5x}$

So,

$$f_2'' - 7f_2' + 10f_2$$

$$= 25e^{5x} - 7(5e^{5x}) + 10(e^{5x})$$

$$= 25e^{5x} - 35e^{5x} + 10e^{5x}$$

$$= 0$$

So, f_2 also solves $y'' - 7y' + 10y = 0$

Since $f_1(x) = e^{2x}$, $f_2(x) = e^{5x}$

are linearly independent solutions to $y''-7y'+10y=0$

We know that every solution to $y''-7y'+10y=0$ is of the form

$$y_h = c_1 e^{2x} + c_2 e^{5x}$$

where c_1, c_2 are any constants

Note: There are infinitely many solutions. Here are two of them:

$c_1 = 1,\ c_2 = -3$: $y_h = e^{2x} - 3e^{5x}$

$c_1 = \frac{1}{2},\ c_2 = \sqrt{2}$: $y_h = \frac{1}{2}e^{2x} + \sqrt{2}e^{5x}$

Step 2: Find a particular solution y_p to

$$y'' - 7y' + 10y = 24e^x$$

Let

$$y_p = 6e^x$$

In topic 8 we learn how to find this

Let's check that it works.

We get:

$$y_p = 6e^x, \quad y_p' = 6e^x, \quad y_p'' = 6e^x$$

So,

$$y_p'' - 7y_p' + 10y_p = 6e^x - 7(6e^x) + 10(6e^x)$$

$$= 6e^x - 42e^x + 60e^x$$
$$= 24e^x$$

Thus, $y_p = 6e^x$ solves
$$y'' - 7y' + 10y = 24e^x.$$

Thus, every solution to
$$y'' - 7y' + 10y = 24e^x$$
is of the form
$$y = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{y_h} + \underbrace{6e^x}_{y_p}$$
where c_1, c_2 are any constants

Ex: Let's solve

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

on $I = (0, \infty)$ ← $\boxed{0 < x}$

Step 1: Find two linearly independent solutions to

$$x^2 y'' - 4xy' + 6y = 0$$

homogeneous equation

Let

$$f_1(x) = x^2$$
$$f_2(x) = x^3$$

topic 10 is where we learn how to find these

Let's plug them in the Wronskian to test for linear independence.

We have:

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

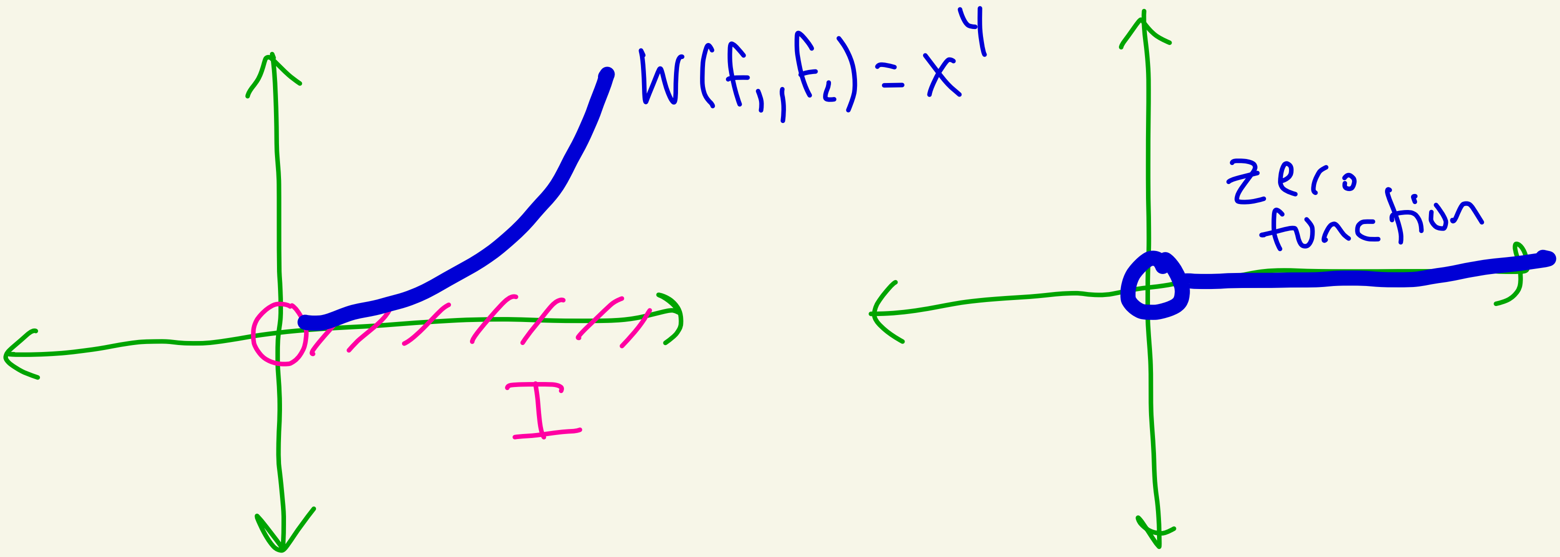
$$= (x^2)(3x^2) - (x^3)(2x)$$

$$= 3x^4 - 2x^4$$

$$= x^4$$

Is this the zero function

on $I = (0, \infty)$?



No, they aren't the same

Thus, $f_1(x) = x^2$, $f_2(x) = x^3$ are linearly independent on $I = (0, \infty)$.

Let's show they solve

$$x^2 y'' - 4xy' + 6y = 0$$

For $f_1 = x^2$ we get:

$$f_1 = x^2, \; f_1' = 2x, \; f_1'' = 2$$

So,

$$x^2 f_1'' - 4x f_1' + 6 f_1$$

$$= x^2(2) - 4x(2x) + 6(x^2)$$

$$= 2x^2 - 8x^2 + 6x^2$$

$$= 0$$

So, f_1 works.

For $f_2 = x^3$ we get:

$$f_2 = x^3, \; f_2' = 3x^2, \; f_2'' = 6x$$

So,

$$x^2 f_2'' - 4x f_2' + 6 f_2$$

$$= x^2(6x) - 4x(3x^2) + 6(x^3)$$

$= 6x^3 - 12x^3 + 6x^3$

$= 0$

So, f_2 works also.

<u>Summary:</u> Since $f_1(x) = x^2$ and $f_2(x) = x^3$ are linearly independent solutions to

$$x^2y'' - 4xy' + 6y = 0$$

every solution is of the form

$$y_h = c_1x^2 + c_2x^3$$

where c_1, c_2 are constants

<u>Step 2:</u> Find a particular solution y_p to

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

on $I = (0, \infty)$

Let

$$y_p = \frac{1}{12x} = \frac{1}{12} x^{-1}$$

topic 9 shows how to find this

Plug it in.

$$y_p = \frac{1}{12} x^{-1}$$

$$y_p' = -\frac{1}{12} x^{-2}$$

$$y_p'' = -\frac{1}{12}(-2x^{-3}) = \frac{1}{6} x^{-3}$$

We have:

$$x^2 y_p'' - 4x y_p' + 6y_p$$

$$= x^2\left(\frac{1}{6}x^{-3}\right) - 4x\left(-\frac{1}{12}x^{-2}\right) + 6\left(\frac{1}{12}x^{-1}\right)$$

$$= \frac{1}{6}x^{-1} + \frac{1}{3}x^{-1} + \frac{1}{2}x^{-1}$$

$$= \left(\frac{1+2+3}{6}\right)x^{-1}$$

$$= x^{-1} = \frac{1}{x}$$

So, $y_p = \frac{1}{12}x^{-1}$ solves

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

Summary: The general solution of

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

on $I = (0, \infty)$ is

$$y = \underbrace{c_1 x^2 + c_2 x^3}_{y_h} + \underbrace{\frac{1}{12} x^{-1}}_{y_p}$$

where c_1, c_2 are any constants