Math 2150-02
9/15/25

Topic 6 – Theory of second order linear equations

A second order linear equation is of the form

$$\underbrace{a_2(x)}\, y'' + \underbrace{a_1(x)}\, y' + \underbrace{a_0(x)}\, y = \underbrace{b(x)}$$

these terms have xs and #s but no ys

To study these equations we need some preliminaries

Def: Let I be an interval. Let f_1 and f_2 be defined on I. We say that f_1 and f_2 are linearly dependent on I if either

① $f_1(x) = c f_2(x)$ for all x in I

or

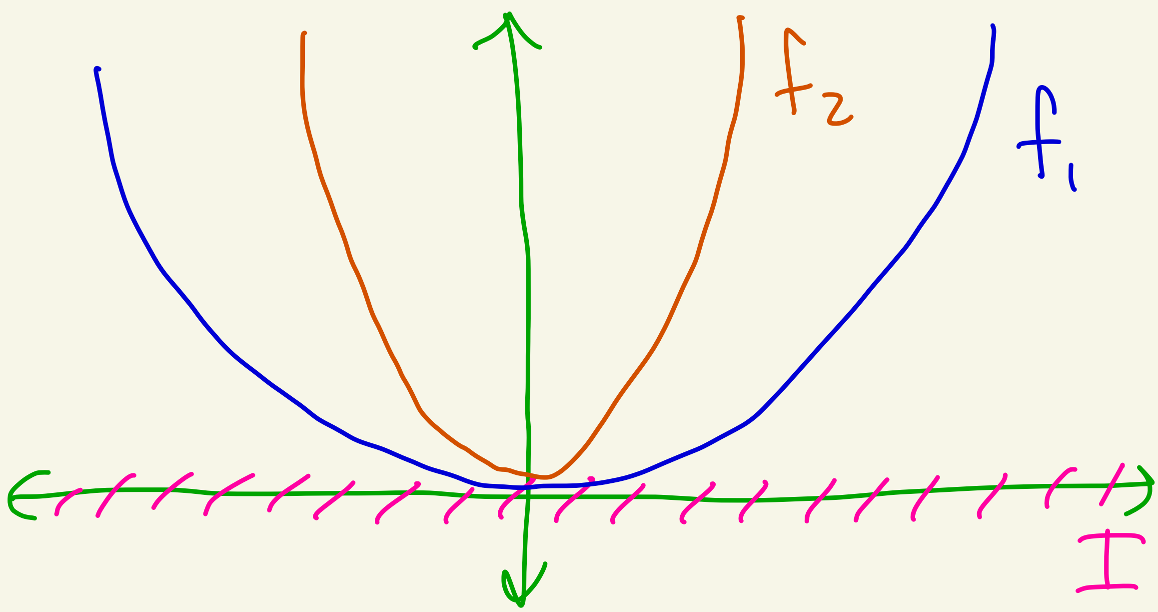
② $f_2(x) = c f_1(x)$ for all x in I

where c is a fixed constant.

If no such constant c exists then f_1 and f_2 are linearly independent.

Ex: Let $I = (-\infty, \infty)$,

$f_1(x) = x^2$, $f_2(x) = 3x^2$.



f_1 and f_2 are linearly dependent on I because

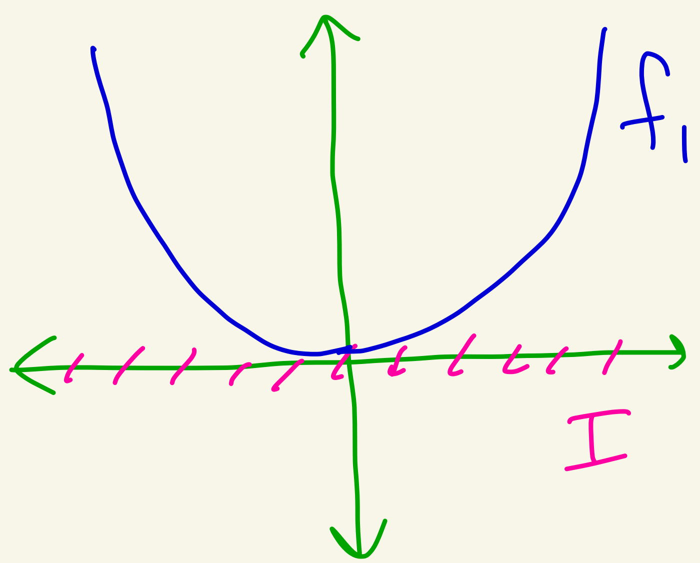
$$f_2(x) = \underbrace{3}_{c=3} \cdot f_1(x)$$

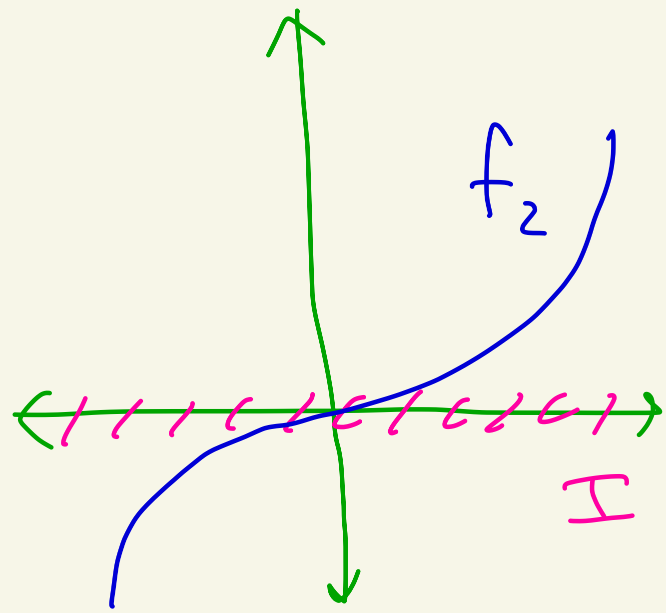
or $f_1 = \frac{1}{3} f_2$

for all x in I

Ex: $I=(-\infty,\infty)$,

$f_1(x)=x^2$, $f_2(x)=x^3$





Are f_1 and f_2 linearly dependent on I?

Suppose $f_1 = c f_2$ on I.

Then, $x^2 = c x^3$ for all x in I.

When $x=1$, then $1^2 = c \cdot 1^3$ and $c=1$.

When $x=2$, then $2^2 = c \cdot 2^3$ and $c = 1/2$.

But $1 \neq 1/2$!

There is no constant c that works.

Similarly there is no constant c with $f_2 = c f_1$.

So, f_1 and f_2 are linearly independent on I.

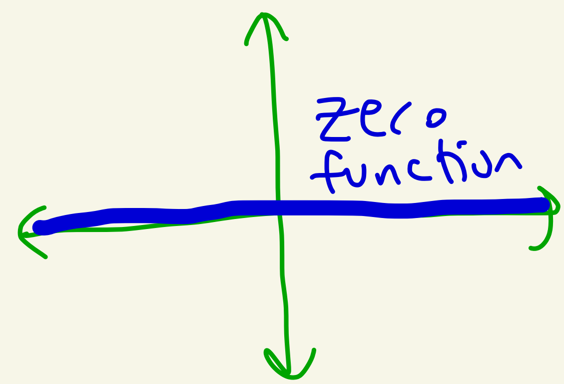
Now we give another way to test for lin. ind. using the Wronskian.

Named after Josef Wronski (1778-1853)

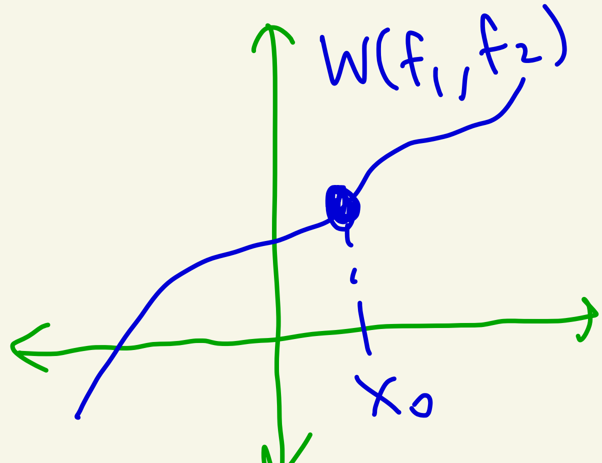
Theorem: Let I be an interval. Let f_1, f_2 be differentiable on I. If the **Wronskian**

$$\underbrace{W(f_1,f_2)}_{\text{notation}} = \underbrace{\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}}_{\text{determinant}} = \underbrace{f_1 f_2' - f_2 f_1'}_{\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}}$$

is not the zero function on I, then f_1 and f_2 are linearly independent.



That is, if there exists x_0 in I where $W(f_1,f_2)(x_0) \neq 0$ then f_1 and f_2 are linearly independent.



Ex: Let $I = (-\infty, \infty)$, $f_1(x) = e^{2x}$, and $f_2(x) = e^{5x}$.
Let's show that f_1 and f_2 are linearly independent on I.

We have

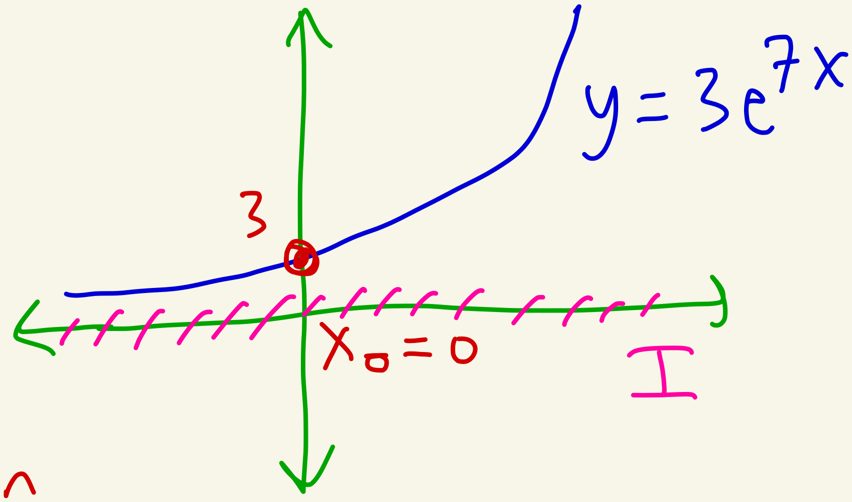
$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{2x} & e^{5x} \\ 2e^{2x} & 5e^{5x} \end{vmatrix}$$

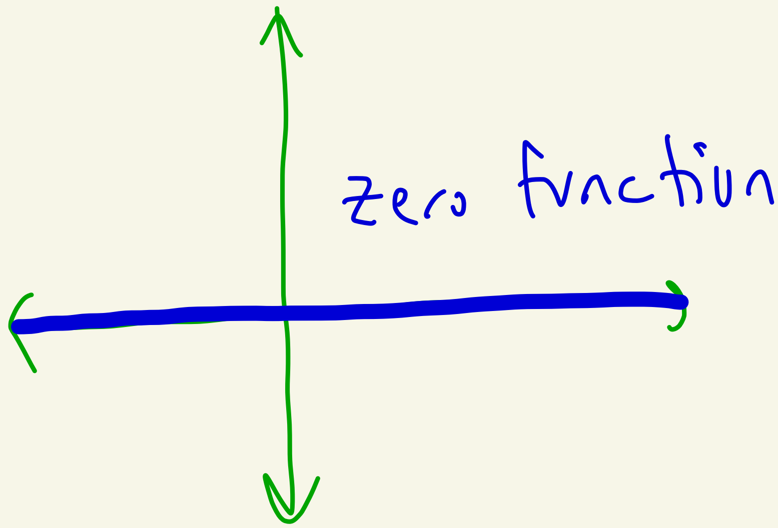
$$= (e^{2x})(5e^{5x}) - (e^{5x})(2e^{2x})$$

$$= 5e^{7x} - 2e^{7x}$$

$$= 3e^{7x}$$



This is <u>not</u> the zero function on $I = (-\infty, \infty)$.



For example, let $x_0 = 0$ then

$$W(f_1, f_2)(0) = 3e^{7(0)} = 3 \neq 0.$$

So, $f_1(x) = e^{2x}$ and $f_2(x) = e^{5x}$ are linearly independent on I.

For the remainder of topic 6 we will be learning the theory of solving

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)}$$

On some interval I where $a_2(x)$, $a_1(x)$, $a_0(x)$, $b(x)$ are all continuous on I and $a_2(x) \neq 0$ on I.

We assume these conditions for the remainder of topic 6.

Ex: $x^2y'' - 4xy + 6y = \frac{1}{x}$

$$I = (0, \infty) \leftarrow \boxed{0 < x}$$

Fact 1: If $f_1(x)$ and $f_2(x)$ are linearly independent solutions to the homogeneous equation

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (*)}$$

homogeneous when $b(x) = 0$

On I, then every solution of $(*)$ on I is of the form

$$\boxed{y_h = c_1 f_1(x) + c_2 f_2(x)}$$

h for homogeneous

where c_1, c_2 are any constants

Fact 2: Suppose we can find a particular solution y_p to

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)} \quad (**)$$

on I, then every solution to $(**)$ is of the form

$$y = \underbrace{c_1 f_1(x) + c_2 f_2(x)}_{\substack{\text{general solution } y_h \text{ to homogeneous equation} \\ a_2(x)y'' + a_1(x)y' + a_0(x)y = 0}} + \underbrace{y_p}_{\text{particular solution to } (**)}$$

Where c_1, c_2 are any constants.