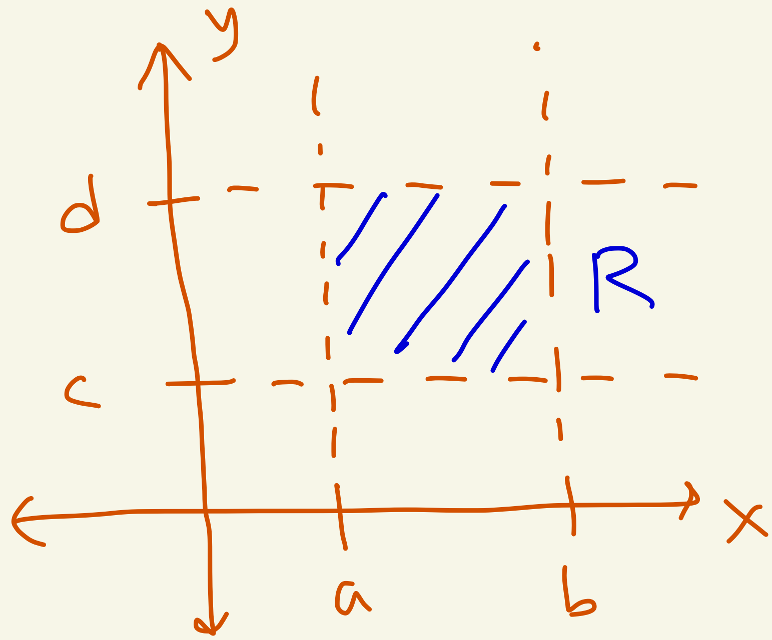
Math 2150-02

9/10/25

Theorem: Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in a rectangle R given by $a<x<b$, $c<y<d$.



Here a,b,c,d can be $\pm\infty$

Then,

$M(x,y)+N(x,y)y'=0$

will be exact if and only if

$$\boxed{\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$$

proof in online notes if interested

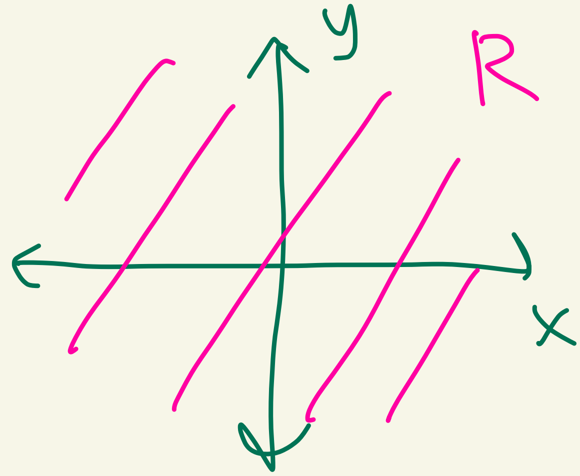
Ex: Consider the equation from last time:

$$2xy + (x^2 - 1)y' = 0$$

We have

$$\left.\begin{aligned} &M(x,y) = 2xy \\ &N(x,y) = x^2 - 1 \\ &\frac{\partial M}{\partial x} = 2y \qquad \frac{\partial N}{\partial x} = 2x \\ &\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial y} = 0 \end{aligned}\right\} \text{Polynomials Continuous everywhere}$$

So in the theorem R would be the xy-plane.



We have $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$.

So, $2xy + (x^2-1)y' = 0$ is exact as we saw last time.

Let's find $f(x,y)$ where the following holds:

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 2xy \quad (1) \\ \frac{\partial f}{\partial y} &= x^2-1 \quad (2) \end{aligned}}$$

$$\frac{\partial f}{\partial x} = M \qquad \frac{\partial f}{\partial y} = N$$

Integrate (1) with respect to x:

$$f(x,y) = x^2 y + \underbrace{D(y)}_{\text{constant with respect to x. Can have \#s and ys}}$$

Integrate ② with respect to y:

$$\boxed{f(x,y) = x^2y - y + \underbrace{E(x)}_{\text{constant w/ respect to y. Can have \#s and xs}}}$$

Set the above equal to get:

$$\cancel{x^2y} + D(y) = \cancel{x^2y} - y + E(x)$$

So,

$$\underbrace{D(y)} = \underbrace{-y} + \underbrace{E(x)}_{0}$$

Set $D(y) = -y$ and $E(x) = 0$.

Plug either one into above

formulas to get f.

For example,

$$f(x,y) = x^2y + D(y)$$
$$= x^2y - y$$

Recall this gives the solution

$$x^2y - y = c \leftarrow \boxed{f(x,y) = c}$$

for

$$2xy + (x^2-1)y' = 0$$

Ex: Find a solution to

$$\boxed{\begin{gathered}(e^x+y)+(2+x+ye^y)y'=0 \\ y(0)=1\end{gathered}}$$

Let's check if the equation is exact.

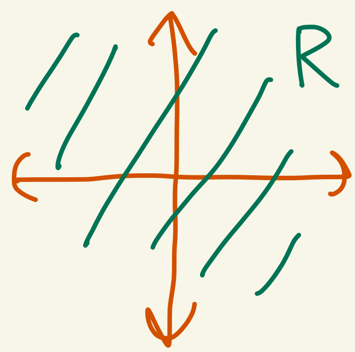
$$M(x,y)=e^x+y$$

$$N(x,y)=2+x+ye^y$$

$$\frac{\partial M}{\partial x}=e^x \qquad \frac{\partial N}{\partial x}=1$$

$$\frac{\partial M}{\partial y}=1 \qquad \frac{\partial N}{\partial y}=e^y+ye^y$$

Continuous everywhere



We have $\frac{\partial M}{\partial y}=1=\frac{\partial N}{\partial x}$.

So we have an exact ODE.
Let's find f where:

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= e^x + y \quad (1) \\ \frac{\partial f}{\partial y} &= 2 + x + ye^y \quad (2) \end{aligned}}$$

$\frac{\partial f}{\partial x} = M$

$\frac{\partial f}{\partial y} = N$

Integrate (1) with respect to x:

$$f(x,y) = e^x + yx + \underbrace{D(y)}_{\text{constant with respect to } x}$$

Integrate (2) with respect to y:

$$f(x,y) = 2y + xy + \underbrace{ye^y - e^y} + \underbrace{E(x)}_{\text{constant with respect to } y}$$

$$\int ye^y\,dy = ye^y - \int e^y\,dy = ye^y - e^y$$

$$u = y \qquad du = dy$$
$$dv = e^y\,dy \qquad v = e^y$$
$$\int u\,dv = uv - \int v\,du$$

Set the above equal to get:

$$e^x + \cancel{yx} + D(y) = 2y + \cancel{xy} + ye^y - e^y + E(x)$$

$$\underbrace{e^x} + \underbrace{D(y)} = \underbrace{2y + ye^y - e^y} + \underbrace{E(x)}$$

Set $D(y) = 2y + ye^y - e^y$

$E(x) = e^x.$

Plug either one in to get f.

For example,

$$f(x,y) = e^x + yx + D(y)$$

$$= e^x + yx + 2y + ye^y - e^y$$

Thus a solution to

$$(e^x + y) + (2 + x + ye^y)y' = 0$$

is given by

$$\underbrace{e^x + yx + 2y + ye^y - e^y = c}_{f(x,y) = c}$$

Where c is any constant.

Now let's find a solution where $y(0)=1$.
Plug in $x=0, y=1$ into the above solution to get:

$$e^0 + (1)(0) + 2(1) + (1)e^1 - e^1 = C$$
$$3 = C$$

Thus,

$$e^x + yx + 2y + ye^y - e^y = 3$$

is a solution to

$$(e^x + y) + (2 + x + ye^y)y' = 0$$
$$y(0) = 1$$