Math 2150-02
8/27/25

Topic 1 continued...

<u>Ex:</u> Show that

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

is a solution to

$$y'' - 4y = 0$$

on $I = (-\infty, \infty)$.

Here c_1, c_2 are any constants

In topic 7 we will see how I got this

<u>Ex:</u> $c_1 = 5$, $c_2 = -\frac{1}{2}$

$$f(x) = 5e^{2x} - \frac{1}{2}e^{-2x}$$

Let's plug f into $y'' - 4y = 0$

and show it solves it.

We get

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

$$f'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x}$$

$$f''(x) = 4c_1 e^{2x} + 4c_2 e^{-2x}$$

note these all exist on $I = (-\infty, \infty)$

Let's plug f into $y'' - 4y = 0$.

We have

$$f'' - 4f$$

$$= (4c_1 e^{2x} + 4c_2 e^{-2x}) - 4(c_1 e^{2x} + c_2 e^{-2x})$$

$$= 0$$

So, f does solve $y'' - 4y = 0$.

Ex: Find c_1, c_2 where

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

solves the initial-value problem

$$y'' - 4y = 0 \leftarrow \text{ODE}$$

$$y'(0) = 0$$

$$y(0) = 1$$

} initial values at $x_0 = 0$

We already know that

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

solves $y'' - 4y = 0$.

Let's make it satisfy

$$f'(0) = 0$$

$$f(0) = 1$$

We have

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$
$$f'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x}$$

So we need

$$1 = c_1 e^{2(0)} + c_2 e^{-2(0)}$$
$$0 = 2c_1 e^{2(0)} - 2c_2 e^{-2(0)}$$

$1 = f(0)$

$0 = f'(0)$

$$1 = c_1 + c_2 \quad (1)$$
$$0 = 2c_1 - 2c_2 \quad (2)$$

(2) gives $c_1 = c_2$.

Plug $c_1 = c_2$ into (1) to get $1 = c_1 + c_1$

Thus, $c_1 = 1/2$

Then, $c_2 = c_1 = 1/2$.

So,

$$f(x) = \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}$$

Topic 3 - First order linear ODEs

We will give a method to solve

$$y' + a(x)y = b(x)$$

on any interval I where $a(x)$ and $b(x)$ are continuous.

Since $a(x)$ is continuous we can find an anti-derivative

$$A(x) = \int a(x)\,dx$$

So, $A'(x) = a(x)$

Multiply $y' + a(x)y = b(x)$

by $e^{A(x)}$ to get:

$$e^{A(x)} y' + e^{A(x)} a(x) y = e^{A(x)} b(x)$$

this side is $\left(e^{A(x)} \cdot y\right)'$

$(fg)' = f'g + fg'$

Check:

$$\left(e^{A(x)} \cdot y\right)' = e^{A(x)} \cdot A'(x) \cdot y + e^{A(x)} \cdot y'$$
$$= e^{A(x)} \cdot a(x) y + e^{A(x)} \cdot y'$$

We get:

$$\left(e^{A(x)} \cdot y\right)' = e^{A(x)} b(x)$$

Integrate both sides to get

$$e^{A(x)} \cdot y = \int e^{A(x)} b(x)\, dx$$

Thus,

$$y = e^{-A(x)} \int e^{A(x)} b(x)\, dx$$

Same as: $\frac{1}{e^{A(x)}}$

Since you can reverse the above process, the above is the general solution to the ODE.

<u>Ex:</u> Solve

$$y' + 2xy = x$$

$b(x) = x$

$a(x) = 2x$

on $I = (-\infty, \infty)$

Let $A(x) = \int 2x\,dx = x^2$ ← don't need $+C$ just need one anti-derivative

multiply $y' + 2xy = x$

by $e^{A(x)} = e^{x^2}$ to get:

$$e^{x^2} y' + e^{x^2} 2xy = xe^{x^2}$$

always equals

$$\left(e^{A(x)} \cdot y\right)'$$

So we get

$$(e^{x^2}\cdot y)' = xe^{x^2}$$

Now integrate:

$$e^{x^2}\cdot y = \int xe^{x^2}\,dx$$

$$\int xe^{x^2}\,dx = \int \tfrac{1}{2}e^{u}\,du$$

$u = x^2$

$du = 2x\,dx$

$\tfrac{1}{2}du = x\,dx$

$$= \tfrac{1}{2}e^{u} + C = \tfrac{1}{2}e^{x^2} + C$$

Thus,
$$e^{x^2} \cdot y = \frac{1}{2} e^{x^2} + C$$

So,
$$y = e^{-x^2}\left(\frac{1}{2} e^{x^2} + C\right)$$

Thus,
$$y = \frac{1}{2} e^{-x^2 + x^2} + C e^{-x^2}$$

$e^0 = 1$

So,
$$y = \frac{1}{2} + C e^{-x^2}$$

Where C is any constant

Ex: Let's solve

$$y' + \cos(x) y = \sin(x)\cos(x)$$

on $I = (-\infty, \infty)$

Let $A(x) = \int \cos(x)\,dx = \sin(x)$

Multiply the above ODE by $e^{A(x)} = e^{\sin(x)}$ to get:

$$e^{\sin(x)} y' + e^{\sin(x)} \cos(x)\, y = e^{\sin(x)} \sin(x)\cos(x)$$

this is always $\left(e^{A(x)} \cdot y\right)'$

We get

$$\left(e^{\sin(x)} \cdot y\right)' = e^{\sin(x)} \sin(x)\cos(x)$$

Integrate to get

$$e^{\sin(x)} \cdot y = \int e^{\sin(x)} \sin(x)\cos(x)\,dx$$

$$\int e^{\sin(x)} \sin(x)\cos(x)\,dx$$

$$= \int e^t \cdot t\,dt = te^t - \int e^t\,dt$$

$t = \sin(x)$

$dt = \cos(x)\,dx$

LIATE

$u = t \qquad du = dt$

$dv = e^t\,dt \qquad v = e^t$

$\int u\,dv = uv - \int v\,du$

$$= te^t - e^t + C$$

$$= \sin(x)e^{\sin(x)} - e^{\sin(x)} + C$$

So,

$$e^{\sin(x)} \cdot y = \sin(x) e^{\sin(x)} - e^{\sin(x)} + C$$

Thus,

$$y = e^{-\sin(x)} \left[\sin(x) e^{\sin(x)} - e^{\sin(x)} + C \right]$$

same as: $\frac{1}{e^{\sin(x)}}$

Therefore,

$$y = \sin(x) - 1 + Ce^{-\sin(x)}$$

where C is any constant.