Math 2150-02
4/9/25

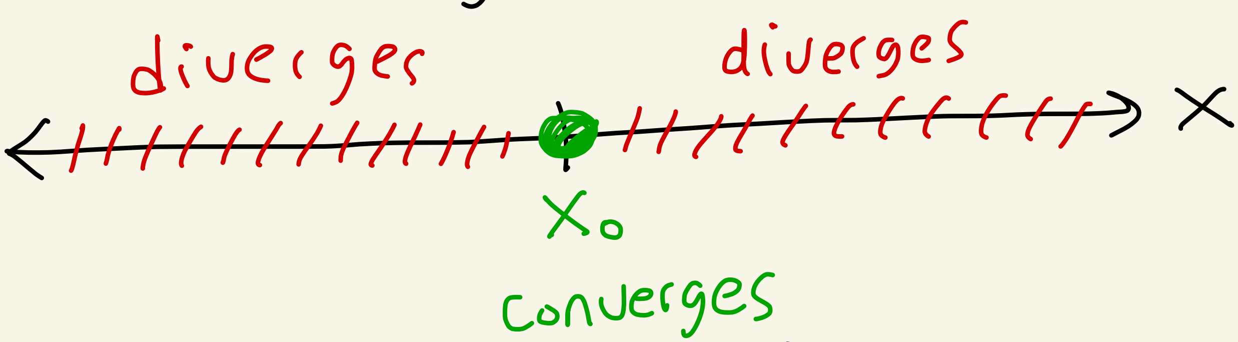
(Topic 11 continued...)

Recall, for a power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

centered at x_0, there are three possibilities:

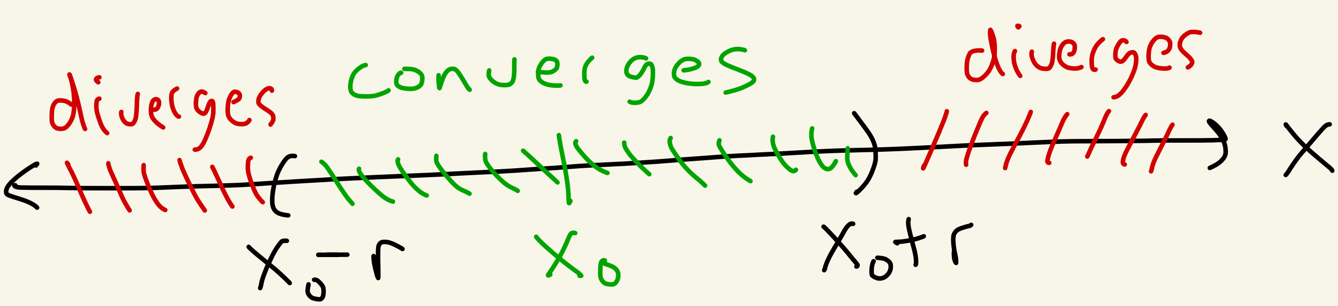
① the series only converges when $x = x_0$



$r = 0$ is the radius of convergence

② there exists $r > 0$ where the series converges when

$$x_0 - r < x < x_0 + r$$



It can converge or diverge at the endpoints, Here r is the radius of convergence

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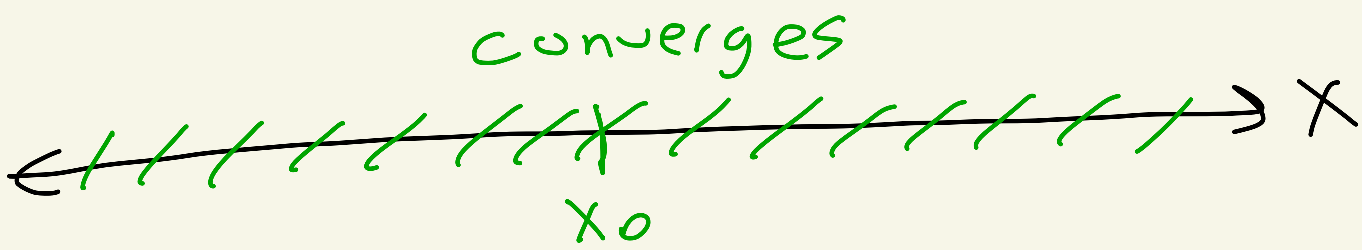
(3) the series converges for all x.



Here $r = \infty$ is the radius of convergence

Ex: Recall

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

Here $x_0 = 0$.

These converge for all x.

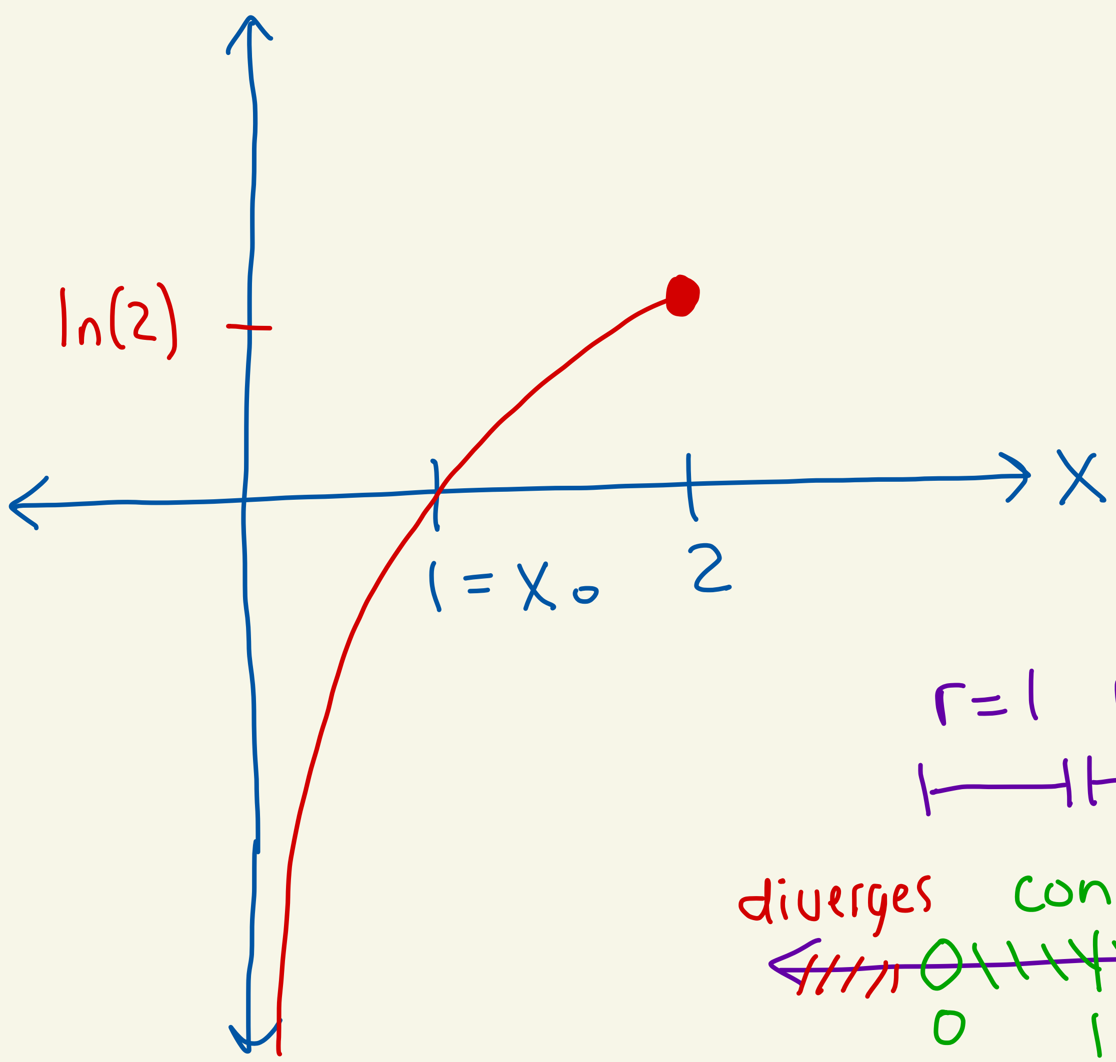
radius of convergence $r = \infty$.

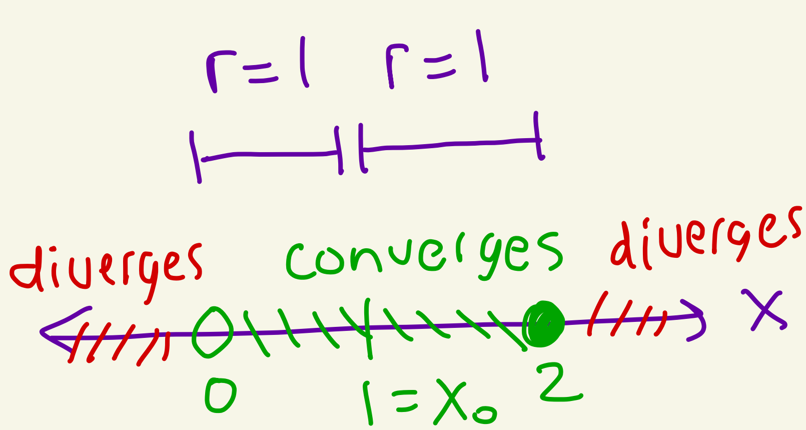
Ex: Recall

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n \quad \leftarrow \boxed{x_0 = 1}$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$$

In calculus you show this converges when $0 < x \leq 2$.





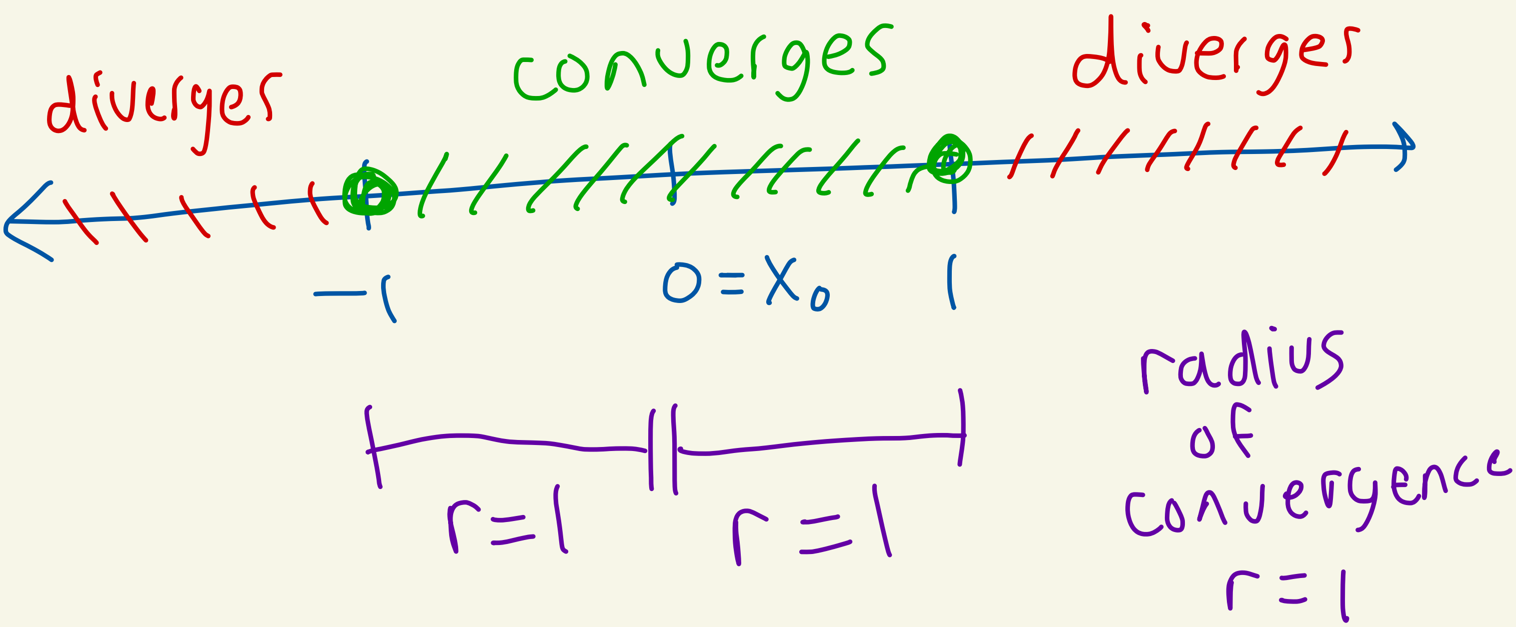
For example

$$\ln\left(\frac{1}{2}\right) = \left(\frac{1}{2}-1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)^2 + \frac{1}{3}\left(\frac{1}{2}-1\right)^3 + \cdots$$
$$= -\frac{1}{2} - \frac{1}{2}\left(-\frac{1}{2}\right)^2 + \frac{1}{3}\left(-\frac{1}{2}\right)^3 - \cdots$$

Ex:

If $-1 \leq x \leq 1$, then

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$
$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$



Fun fact (Approximate π)

$$\frac{\pi}{4} = \tan^{-1}(1)$$
$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

So,
$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \cdots$$

Theorem: (Taylor series)

If
$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

has radius of convergence $r > 0$

then
$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

Ex: Find a power series $f(x) = x^2$ centered at $x_0 = 2$.

We have

$$\boxed{f^{(0)}(x) = x^2}$$

$$f(x) = x^2$$

$$f'(x) = 2x$$

$$f''(x) = 2$$

$$f^{(3)}(x) = 0$$

$$\vdots \qquad \vdots$$

0 from here on

So,

$$x^2 = \frac{2^2}{0!} + \frac{2(2)}{1!}(x-2)^1 + \frac{2}{2!}(x-2)^2$$

$$f(x) = \frac{f^{(0)}(2)}{0!} + \frac{f'(2)}{1!}(x-2)^1 + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^{(4)}(2)}{3!}(x-2)^4$$

$$\underbrace{+ \cdots}_{\text{all zero}}$$

Thus,

$$\boxed{x^2 = 4 + 4(x-2) + 1 \cdot (x-2)^2}$$

This converges for all x since its a finite sum.

So, $r = \infty$

Theorem: If

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

has radius of convergence $r > 0$

then

$$f'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \cdots$$

and

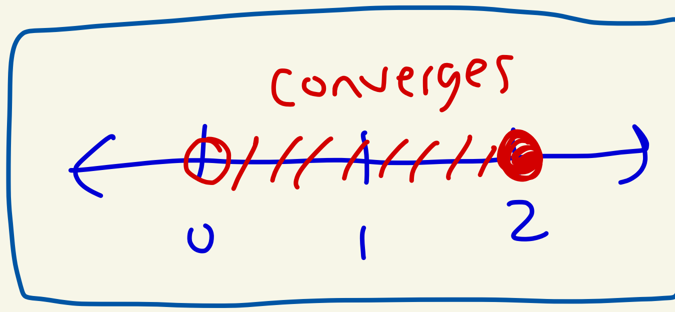
$$\int f(x)\,dx = a_0(x-x_0) + \frac{a_1}{2}(x-x_0)^2 + \frac{a_2}{3}(x-x_0)^3 + \cdots$$

and these both will also have radius of convergence r.

Ex:

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

has radius of convergence $r=1$.



Then,

$$\frac{1}{x} = \frac{d}{dx}\ln(x)$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$

has radius of convergence $r=1$.