

Math 2150-02
4/9/25



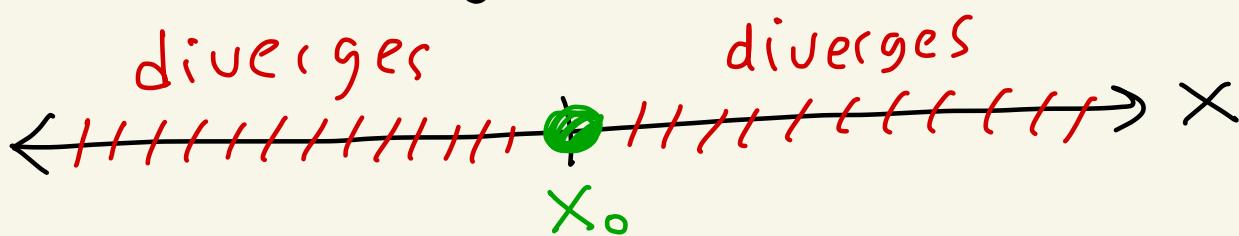
(Topic 11 continued...)

Recall, for a power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \dots$$

centered at x_0 , there are three possibilities:

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- ① the series only converges when $x = x_0$



$r=0$ is the radius of convergence

- ② there exists $r > 0$ where the series converges when

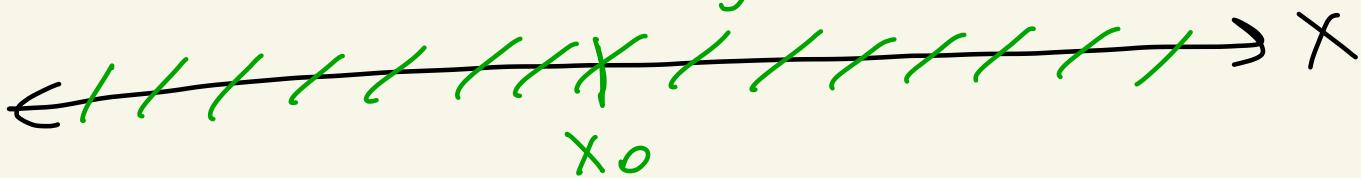
$$x_0 - r < x < x_0 + r$$



It can converge or diverge at the endpoints. Here r is the radius of convergence

③ the series converges for all x .

converges



Here $r = \infty$ is the radius of convergence

Ex: Recall

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

Here $x_0 = 0$.

These converge for all x .

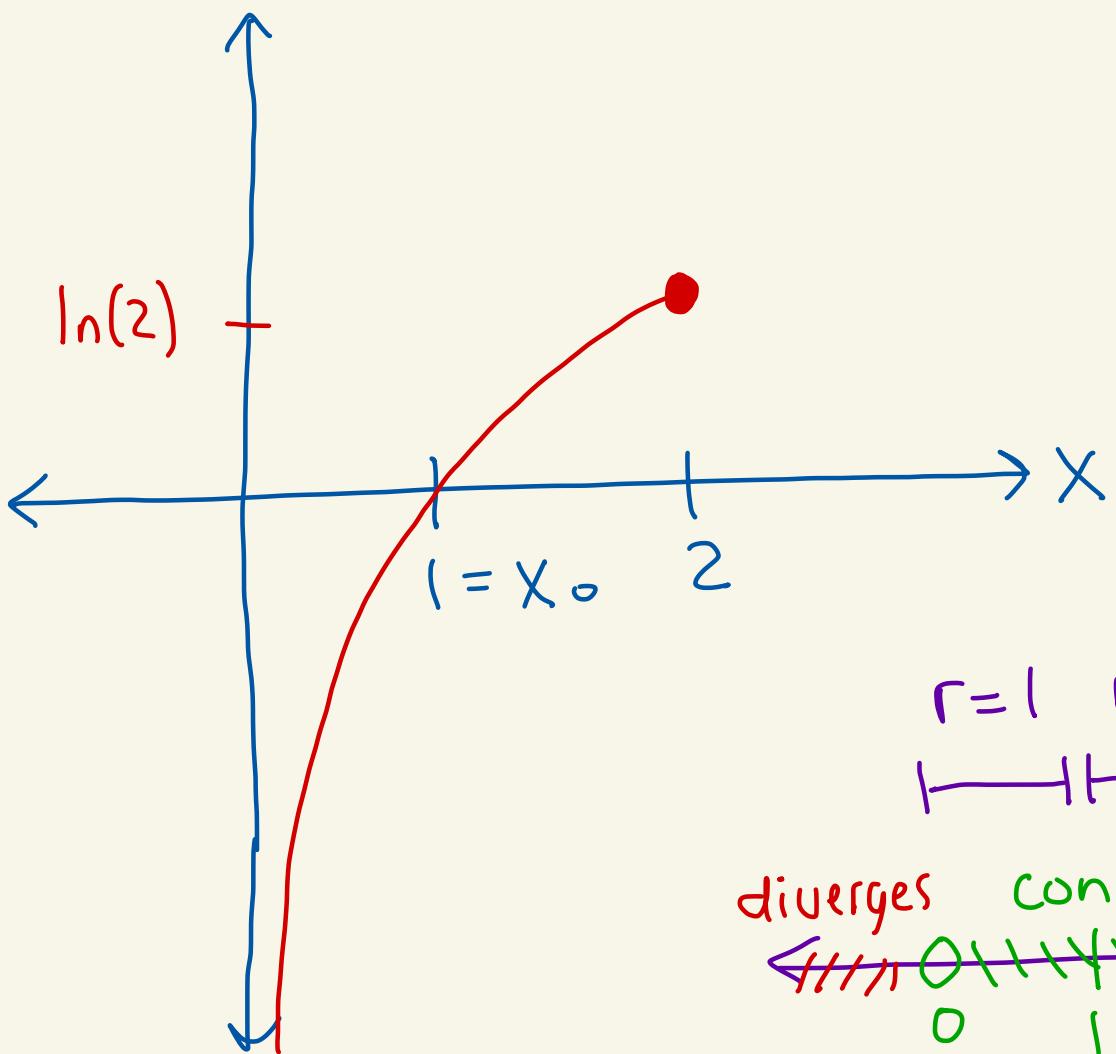
Radius of convergence $r = \infty$.

Ex: Recall

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n \quad \leftarrow x_0 = 1$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots$$

In calculus you show this converges when $0 < x \leq 2$.



$$r=1 \quad r=1$$

$\ln(x)$ diverges at $x=0$ and $x=2$. The interval $(1, 2)$ where the series converges is highlighted with a green oval around the point $x=1$ and a green arrow pointing to the right along the x-axis, with the text 'converges' written above it. The regions outside this interval are marked with red diagonal lines and the text 'diverges' written below them, with the text 'diverges' written above the left boundary and 'diverges' written above the right boundary.

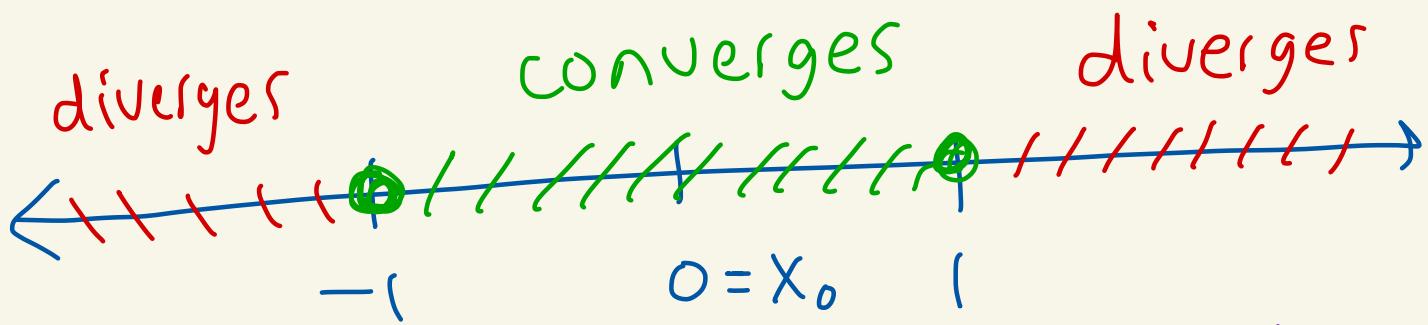
For example

$$\ln\left(\frac{1}{z}\right) = \left(\frac{1}{z}-1\right) - \frac{1}{2}\left(\frac{1}{z}-1\right)^2 + \frac{1}{3}\left(\frac{1}{z}-1\right)^3 + \dots$$
$$= -\frac{1}{2} - \frac{1}{2}\left(-\frac{1}{z}\right)^2 + \frac{1}{3}\left(-\frac{1}{z}\right)^3 - \dots$$

Ex:

If $-1 \leq x \leq 1$, then

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$
$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$$



$$r=1 \quad r=1$$

radius
of
convergence
 $r=1$

Fun fact (Approximate π)

$$\frac{\pi}{4} = \tan^{-1}(1) \\ = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

So,

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \dots$$

Theorem: (Taylor Series)

If

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

has radius of convergence $r > 0$

then

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

Ex: Find a power series

$f(x) = x^2$ centered at $x_0 = 2$.

We have

$$f(x) = x^2$$

$$f^{(0)}(x) = x^2$$

$$f'(x) = 2x$$

$$f''(x) = 2$$

$$f^{(3)}(x) = 0$$

$$\begin{matrix} \vdots & \vdots \\ \vdots & \vdots \end{matrix}$$

0 from
here
on

$$\begin{aligned} f(x) &= \underbrace{\frac{x^2}{0!}}_{f^{(0)}(2)} + \underbrace{\frac{f'(2)}{1!}(x-2)^1}_{f'(z)} + \underbrace{\frac{f''(2)}{2!}(x-2)^2}_{f''(z)} \\ &+ \underbrace{\frac{f'''(2)}{3!}(x-2)^3}_{f'''(z)} + \underbrace{\frac{f^{(4)}(2)}{4!}(x-2)^4}_{f^{(4)}(z)} \end{aligned}$$

$\underbrace{\quad}_{\text{all zero}} + \dots$

Thus,

$$x^2 = 4 + 4(x-2) + 1 \cdot (x-2)^2$$

This converges for all x
since its a finite sum.

So, $r = \infty$

Theorem: If

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \dots$$

has radius of convergence $r > 0$

then

$$f'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \dots$$

and

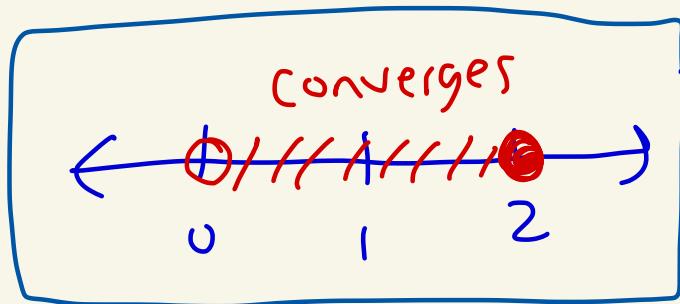
$$\int f(x) dx = a_0(x-x_0) + \frac{a_1}{2}(x-x_0)^2 + \frac{a_2}{3}(x-x_0)^3 + \dots$$

and these both will also have
radius of convergence r .

Ex:

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots$$

has radius of convergence $r=1$.



Then,

$$\frac{1}{x} = \frac{d}{dx} \ln(x)$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots$$

has radius of convergence $r=1$.

