Math 2150-02

12/3/25

Test 2 - #3

Given: $y_1 = x^2$ solves
$$x^2 y'' + 2xy' - 6y = 0$$
on $I = (0, \infty)$
Find general solution

Divide by x^2 to get:
$$y'' + \boxed{\frac{2}{x}} y' - \frac{6}{x^2} y = 0$$

$$\boxed{a_1 = \frac{2}{x}}$$

$$y_2 = y_1 \cdot \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx$$

$$= x^2 \int \frac{e^{-\int \frac{2}{x} dx}}{(x^2)^2} \, dx$$

$$= x^2 \int \frac{e^{-2\int \frac{1}{x} dx}}{x^4}$$

$$= x^2 \int \frac{e^{-2\ln(x)}}{x^4} \, dx$$

$$= x^2 \int \frac{\cancel{e^{\ln}}(x^{-2})}{x^4} \, dx$$

$$= x^2 \int \frac{x^{-2}}{x^4} \, dx$$

$$= x^2 \int x^{-6}\,dx$$

$$= x^2 \left(\frac{x^{-6+1}}{-6+1} \right)$$

$$= \boxed{-\frac{1}{5}x^{-3}} \leftarrow y_2$$

Answer:

$$\boxed{\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 x^2 + c_2\left(-\frac{1}{5}x^{-3}\right) \end{aligned}}$$

Test 2 – #5

Find the first four non-zero terms to a power series solution to

$$y'' - 2xy' - y = 0$$

$$y'(0) = 1, \quad y(0) = 1$$

center $x_0 = 0$

What's the radius of convergence of your answer?

Since the coefficients of the ODE are polynomials the radius of convergence will be $r = \infty$ for our answer.

Our answer will have the form:

$$y(x) = y(0) + y'(0)(x-0) + \frac{y''(0)}{2!}(x-0)^2 + \frac{y'''(0)}{3!}(x-0)^3 + \cdots$$

$$= \overset{1}{\boxed{y(0)}} + \overset{1}{\boxed{y'(0)}}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

Let's find $y''(0)$.

$$y'' - 2xy' - y = 0$$

$$y'' = 2xy' + y$$

$$y''(0) = 2(0)[y'(0)] + [y(0)]$$

$$= 2(0)[1] + [1]$$

$$= 0 + 1 = 1$$

So, $\boxed{y''(0) = 1}$

Let's find $y'''(0)$.

$$y'' = \underbrace{2xy'}_{} + \underbrace{y}_{}$$

$$y''' = \underbrace{2y' + 2xy''}_{} + \underbrace{y'}_{}$$

$$y''' = 3y' + 2xy''$$

$$y'''(0) = 3\left[y'(0)\right] + 2(0)\left[y''(0)\right]$$

$$= 3\left[1\right] + 2(0)\left[1\right]$$

$$= 3 + 0 = 3$$

$$y'''(0) = 3$$

So,

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

$$= 1 + 1 \cdot x + \frac{1}{2!}x^2 + \frac{3}{3!}x^3 + \cdots$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \cdots$$

HW 8 #1(c)

Given that $y_h = c_1 e^{-2x} + c_2 x e^{-2x}$ for $y'' + 4y' + 4y = 0$,

find y_p for

$y'' + 4y' + 4y = 4x^2 - 8x$

and state the general solution.

Guess:

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Plug in to get

$$(2A) + 4(2Ax + B) + 4(Ax^2 + Bx + c) = 4x^2 - 8x$$

$$2A + 8Ax + 4B + 4Ax^2 + 4Bx + 4C = 4x^2 - 8x$$

$$\boxed{4A}x^2 + \boxed{(8A+4B)}x + \underbrace{\boxed{(2A+4B+4c)}}_{0} = \boxed{4}x^2 - 8x$$

$$4A = 4$$
$$8A + 4B = -8$$
$$2A + 4B + 4C = 0$$

$$\boxed{A = 1}$$

$$8(1) + 4B = -8$$
$$B = -4$$

$$2(1) + 4(-4) + 4C = 0$$
$$4C = 14$$
$$C = 7/2$$

So, $y_p = Ax^2 + Bx + C$

$$= x^2 - 4x + 7/2$$

General Sol.

$$y = y_h + y_p = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 - 4x + \frac{7}{2}$$

Test 1 #3

Solve the linear equation

$$y' + \boxed{\frac{1}{x}} y = 3x^2 - \frac{1}{x}$$

on $I = (0, \infty)$

$$A(x) = \int \frac{1}{x}\,dx = \ln(x)$$

$$e^{A(x)} = \cancel{e}^{\cancel{\ln}(x)} = x$$

multiply eqn. by x to get:

$$xy' + y = 3x^3 - 1$$

$$(xy)' = 3x^3 - 1$$

$$xy = 3\left(\frac{x^4}{4}\right) - x + C$$

$$\boxed{y = \frac{3}{4}x^3 - 1 + \frac{C}{x}}$$

$$\boxed{\text{Test 2 - \#2}}$$

Solve

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

given that

$$y_h = c_1 \underbrace{\boxed{e^{2x}}}_{y_1} + c_2 \underbrace{\boxed{xe^{2x}}}_{y_2}$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 1 \cdot e^{2x} + x \cdot 2e^{2x} \end{vmatrix}$$

$$= e^{2x}\left(e^{2x} + 2xe^{2x}\right)$$

$$- 2e^{2x}\left(xe^{2x}\right)$$

$$= e^{4x} + \cancel{2xe^{4x}} - \cancel{2xe^{4x}}$$

$$= e^{4x}$$

$$v_1 = \int \frac{-y_2\, b(x)}{W(y_1, y_2)}\, dx$$

$$= \int \frac{-xe^{2x}(x+1)e^{2x}}{e^{4x}}\, dx$$

$$= \int \frac{-x(x+1)\cancel{e^{4x}}}{\cancel{e^{4x}}}\, dx$$

$$= \int (-x^2 - x)\,dx = -\frac{1}{3}x^3 - \frac{1}{2}x^2$$

$$v_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)}\,dx = \int \frac{e^{2x}(x+1)e^{2x}}{e^{4x}}\,dx$$

$$= \int \frac{(x+1)\cancel{e^{4x}}}{\cancel{e^{4x}}}\,dx$$

$$= \int (x+1)\,dx = \frac{1}{2}x^2 + x$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x}$$

Gen. sol.

$$y = y_h + y_p = \cdots$$