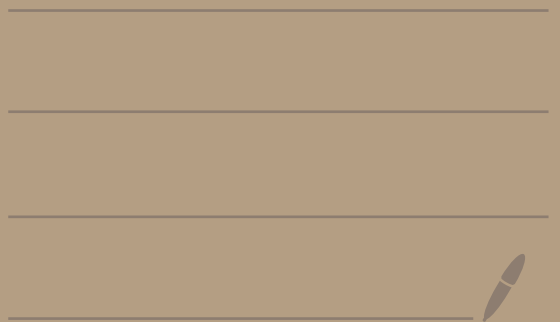


Math 2150-02

11/5/25



(topic 12 continued...)

Ex: Consider

$$y' - 2xy = 0$$

$$y(0) = 1$$

$x_0 = 0$
center of
power series

Let's find a power series solution!

coefficients

$$\begin{matrix} -2x \\ 0 \end{matrix}$$

} polynomials, analytic everywhere
with $r = \infty$

So the solution we find will
have radius of convergence $r = \infty$

We will expand our answer
around $x_0 = 0$. It will look like:

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \dots$$

Given: $y(0) = 1$

$$y' - 2xy = 0$$

$$y' = 2xy$$

plug
in
 $x=0$

$$y'(0) = 2(0)[y(0)] = 2(0)(1) = 0$$

$$y'(0) = 0$$

Now differentiate $y' = 2xy$
with respect to x to get:

$$y'' = 2y + 2xy'$$

$$\begin{aligned}y''(0) &= 2[y(0)] + 2(0)[y'(0)] \\&= 2[1] + 2(0)[0] \\&= 2\end{aligned}$$

$$y''(0) = 2$$

Differentiate $y'' = 2y + 2xy'$
with respect to x to get:

$$y''' = \underline{2y'} + 2y' + 2xy''$$

$$y''' = \underline{4y'} + 2xy''$$

$$y'''(0) = 4 \underbrace{[y'(0)]}_0 + 2(0) \underbrace{[y''(0)]}_2$$

$$y'''(0) = 0$$

Differentiate $y''' = 4y' + 2xy''$
to get:

$$y'''' = 4y'' + 2y'' + 2xy'''$$

$$y'''' = 6y'' + 2xy'''$$

$$y''''(0) = 6 \underbrace{[y''(0)]}_2 + 2(0) \underbrace{[y'''(0)]}_0$$

$$y''''(0) = 12$$

We get

$$y(x) = y(0) + \frac{y'(0)}{1!} x + \frac{y''(0)}{2!} x^2 + \frac{y'''(0)}{3!} x^3 + \frac{y^{(4)}(0)}{4!} x^4 + \dots$$

$$= 1 + \frac{0}{1!} x + \frac{2}{2!} x^2 + \frac{0}{3!} x^3 + \frac{12}{4!} x^4 + \dots$$

So,

$$y(x) = 1 + x^2 + \frac{1}{2} x^4 + \dots$$

with radius of convergence $r = \infty$

Ex: Consider

$$y'' + x^2 y' - (x-1)y = \ln(x)$$

$$y'(1) = 0$$

$$y(1) = 0$$

$$x_0 = 1$$

Find a power series solution
and its radius of convergence.

We expand around $x_0 = 1$.

coefficients

$$\left. \begin{array}{l} x^2 \\ -(x-1) \end{array} \right]$$

polynomials, analytic at $x_0 = 1$
radius of convergence $r = \infty$

$$\ln(x) = (x-1) + \frac{1}{2}(x-1)^2 + \dots \quad] \quad r = 1$$

The minimum of ∞, ∞ , and 1 is 1.

So our solution will have radius of convergence at least $r=1$.

Let's find the solution.

Given:

$$y(1) = 0$$

$$y'(1) = 0$$

$$y'' + x^2 y' - (x-1)y = \ln(x)$$

$$y'' = \ln(x) - x^2 y' + (x-1)y$$

Plug in $x=1$:

$$y''(1) = \underbrace{\ln(1)}_0 - (1)^2 \underbrace{[y'(1)]}_0 + \underbrace{(1-1)}_0 \underbrace{[y(1)]}_0$$

$$y''(1) = 0$$

Differentiate

$$y'' = \ln(x) - \underbrace{x^2 y'} + \underbrace{(x-1)y}$$

with respect to x to get:

$$y''' = \frac{1}{x} - \underbrace{2xy'} - \underbrace{x^2 y''} + (1)y + (x-1)y'$$

plug in $x=1$:

$$y'''(1) = \frac{1}{1} - 2(1) \overbrace{[y'(1)]}^0 - (1)^2 \overbrace{[y''(1)]}^0 + (1) \underbrace{[y(1)]}_0 + \underbrace{(1-1)}_0 \underbrace{[y'(1)]}_0$$

$$y'''(1) = 1$$

If you differentiated again
you would get $y'''(1) = -3$

So,

$$y(x) = \overbrace{y(1)}^0 + \overbrace{\frac{y'(1)}{1!}}^0 (x-1) + \overbrace{\frac{y''(1)}{2!}}^0 (x-1)^2 \\ + \underbrace{\frac{y'''(1)}{3!}}_{1/3!} (x-1)^3 + \underbrace{\frac{y^{(4)}(1)}{4!}}_{-3/4!} (x-1)^4 + \dots$$

Thus,

$$y(x) = \frac{1}{6} (x-1)^3 - \frac{1}{8} (x-1)^4 + \dots$$

With radius of convergence is
at least $r=1$.

converges at least here

