Math 2150-02
11/5/25

(topic 12 continued...)

<u>Ex:</u> Consider

$$\boxed{\begin{aligned} y' - 2xy &= 0 \\ y(0) &= 1 \end{aligned}}$$

$x_0 = 0$
center of power series

Let's find a power series solution!

<u>coefficients</u>

$$\left.\begin{array}{c} -2x \\ 0 \end{array}\right\}$$ polynomials, analytic everywhere with $r = \infty$

So the solution we find will have radius of convergence $r = \infty$

We will expand our answer around $x_0 = 0$. It will look like:

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y''''(0)}{4!}x^4 + \dots$$

Given: $\boxed{y(0) = 1}$

$$y' - 2xy = 0$$

plug in $x = 0$

$$y' = 2xy$$

$$y'(0) = 2(0)[y(0)] = 2(0)(1) = 0$$

$$\boxed{y'(0) = 0}$$

Now differentiate $y' = 2xy$ with respect to x to get:

$$\boxed{y'' = 2y + 2xy'}$$

$$y''(0) = 2[y(0)] + 2(0)[y'(0)]$$
$$= 2[1] + 2(0)[0]$$
$$= 2$$
$$\boxed{y''(0) = 2}$$

Differentiate $y'' = 2y + 2xy'$ with respect to x to get:

$$y''' = 2y' + 2y' + 2xy''$$
$$y''' = 4y' + 2xy''$$
$$y'''(0) = 4\underbrace{[y'(0)]}_{0} + 2(0)\underbrace{[y''(0)]}_{2}$$

$$\boxed{y'''(0) = 0}$$

Differentiate $y''' = 4y' + 2xy''$ to get:

$$y'''' = 4y'' + 2y'' + 2xy'''$$

$$y'''' = 6y'' + 2xy'''$$

$$y''''(0) = 6\underbrace{\left[y''(0)\right]}_{2} + 2(0)\underbrace{\left[y'''(0)\right]}_{0}$$

$$\boxed{y''''(0) = 12}$$

We get

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y''''(0)}{4!}x^4 + \dots$$

$$= 1 + \frac{0}{1!}x + \frac{2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{12}{4!}x^4 + \dots$$

So,

$$y(x) = 1 + x^2 + \frac{1}{2}x^4 + \dots$$

with radius of convergence $r = \infty$

Ex: Consider

$$\boxed{\begin{gathered} y'' + x^2 y' - (x-1)y = \ln(x) \\ y'(1) = 0 \\ y(1) = 0 \end{gathered}}$$

$x_0 = 1$

Find a power series solution and it's radius of convergence.

We expand around $x_0 = 1$.

coefficients

x^2

$-(x-1)$

] polynomials, analytic at $x_0 = 1$
radius of convergence $r = \infty$

$\ln(x) = (x-1) + \frac{1}{2}(x-1)^2 + \dots$] $r = 1$

The minimum of ∞, ∞, and 1 is 1.

So our solution will have radius of convergence at least $r=1$.

Let's find the solution.

<u>Given:</u>

$$\boxed{\begin{aligned} y(1) &= 0 \\ y'(1) &= 0 \end{aligned}}$$

$$y'' + x^2 y' - (x-1)y = \ln(x)$$

$$y'' = \ln(x) - x^2 y' + (x-1)y$$

Plug in $x=1$:

$$y''(1) = \underbrace{\ln(1)}_{0} - (1)^2 \underbrace{[y'(1)]}_{0} + \underbrace{(1-1)}_{0}\underbrace{[y(1)]}_{0}$$

$$\boxed{y''(1) = 0}$$

Differentiate

$$y'' = \ln(x) - x^2 y' + (x-1)y$$

with respect to x to get:

$$y''' = \frac{1}{x} - 2xy' - x^2 y'' + (1)y + (x-1)y'$$

Plug in $x=1$:

$$y'''(1) = \frac{1}{1} - 2(1)\overbrace{[y'(1)]}^{0} - (1)^2 \overbrace{[y''(1)]}^{0}$$

$$+ (1)\underbrace{[y(1)]}_{0} + \underbrace{(1-1)}_{0}\underbrace{[y'(1)]}_{0}$$

$$\boxed{y'''(1) = 1}$$

If you differentiated again you would get $\boxed{y''''(1) = -3}$

So,

$$y(x) = \overbrace{y(1)}^{0} + \overbrace{\frac{y'(1)}{1!}}^{0}(x-1) + \overbrace{\frac{y''(1)}{2!}}^{0}(x-1)^2$$

$$+ \underbrace{\frac{y'''(1)}{3!}}_{1/3!}(x-1)^3 + \underbrace{\frac{y''''(1)}{4!}}_{-3/4!}(x-1)^4 + \cdots$$

Thus,

$$y(x) = \frac{1}{6}(x-1)^3 - \frac{1}{8}(x-1)^4 + \cdots$$

with radius of convergence is at least $r = 1$.