Math 2150-02

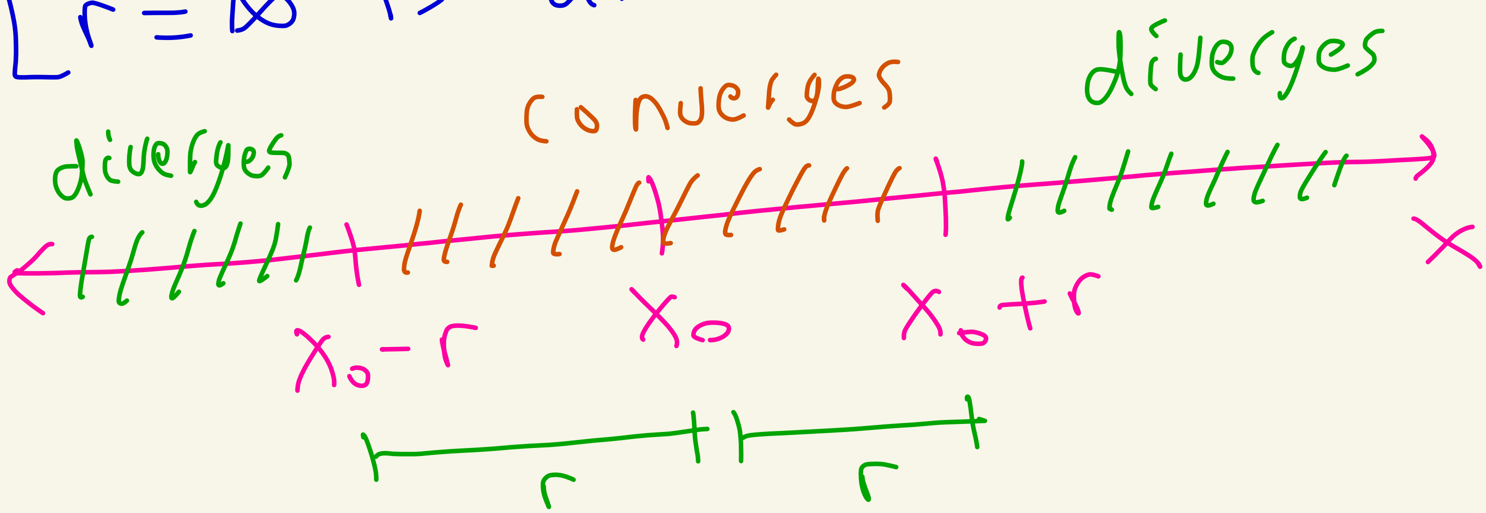
11/3/25

Topic 12 - Power series solutions to ODEs

<u>Def:</u> We say that a function $f(x)$ is <u>analytic at x_0</u> if it has a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

centered at x_0 with positive radius of convergence $r > 0$
[$r = \infty$ is allowed]

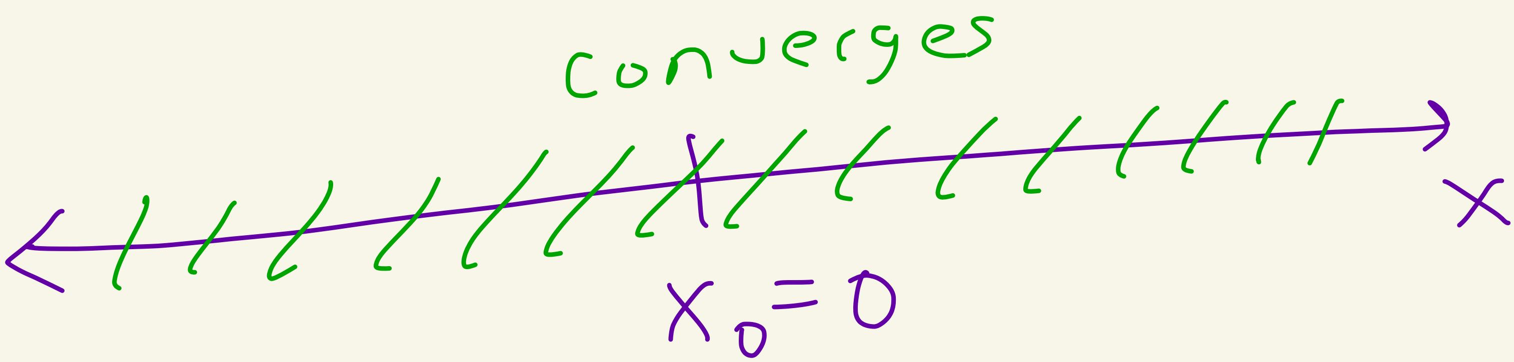


<u>Ex:</u> Is $f(x) = \sin(x)$ analytic at $x_0 = 0$?

Yes.

$$\sin(x) = x \underbrace{- \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}_{\text{power series for } \sin(x) \text{ centered at } x_0 = 0}$$

radius of convergence $r = \infty$

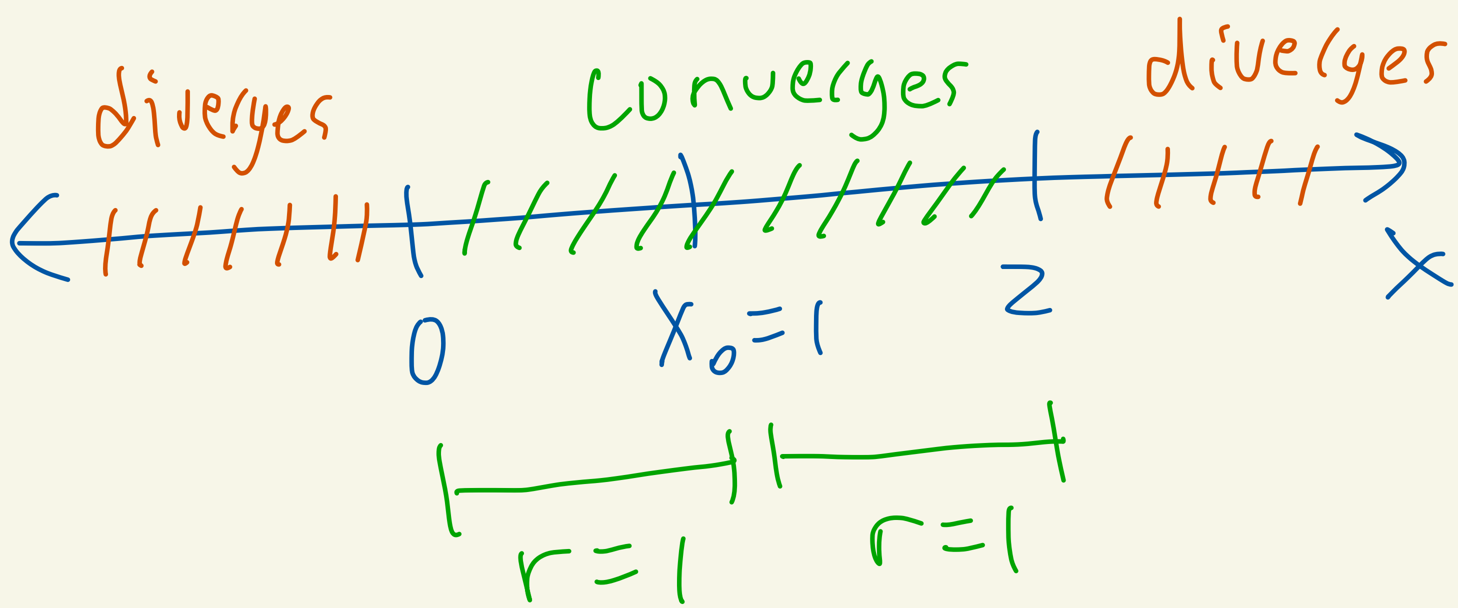


Ex: Is $f(x) = \frac{1}{x}$ analytic at $x_0 = 1$?

Yes.

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$

radius of convergence $r = 1$

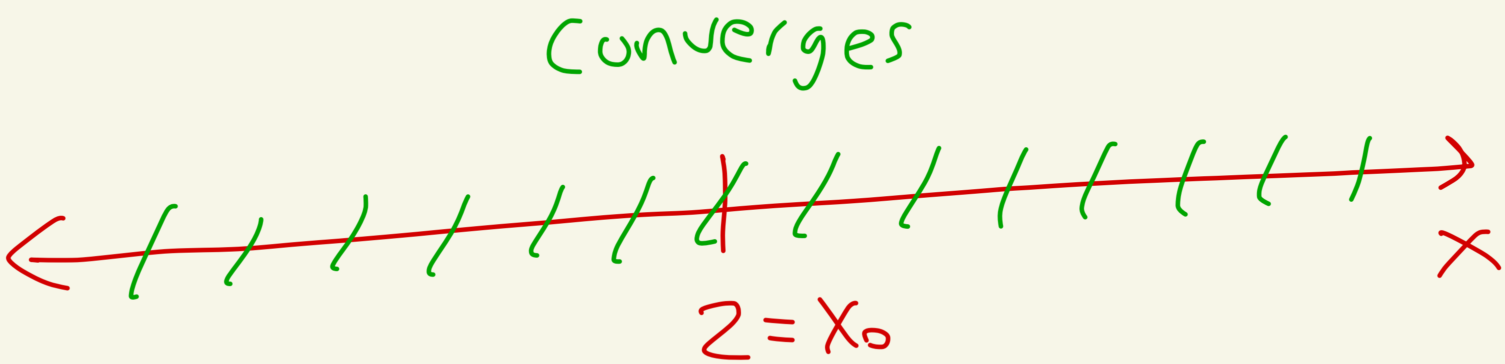


Ex: Is $f(x) = x^2$ analytic at $x_0 = 2$?

Yes.

$$x^2 = 4 + 4(x-2) + (x-2)^2$$

radius of convergence $r = \infty$



Facts:

- polynomials are analytic at every x_0
- e^x, $\sin(x)$, $\cos(x)$ are analytic at every x_0
- rational functions (ratio of polynomials) are analytic at every x_0 except possibly where the denominator is zero

Ex: $3x^5 + x^3 - 1$ ← polynomial

is analytic at every x_0.

Ex: $f(x) = \dfrac{x}{x^2-1}$ ← rational function

$\dfrac{x}{x^2-1}$ is analytic at every $\underbrace{x_0 \neq \pm 1}$

$x^2 - 1 = 0$
$x = \pm 1$

For example, suppose $x_0 = 0$.
We have

$$\frac{x}{x^2-1} = \frac{x}{-1}\left[\frac{1}{1-x^2}\right]$$

$$\boxed{\frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 + \cdots \quad -1 < u < 1}$$

$$u = x^2 \rightarrow -1 < x^2 < 1$$
$$\rightarrow -1 < x < 1$$

$$= -x\left[1 + x^2 + (x^2)^2 + (x^2)^3 + \cdots\right]$$

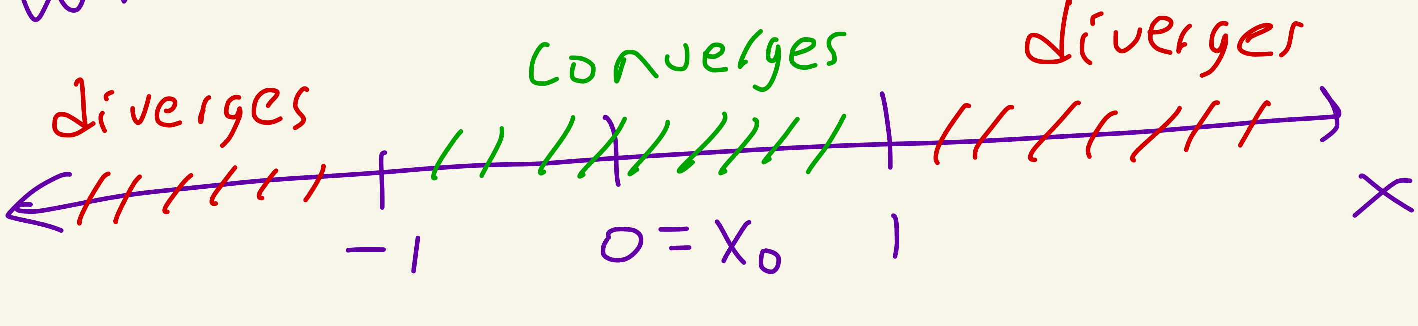
$$= -x\left[1 + x^2 + x^4 + x^6 + \cdots\right]$$

$$= -x - x^3 - x^5 - x^7 - \cdots$$

So,

$$\frac{x}{x^2 - 1} = -x - x^3 - x^5 - x^7 - \cdots$$

when $-1 < x < 1$



Theorem

Consider either of the initial-value problems:

$$\boxed{\begin{array}{c} y' + a_0(x) y = b(x) \\ y(x_0) = y_0 \end{array}}$$

first order

OR

$$\boxed{\begin{array}{c} y'' + a_1(x) y' + a_0(x) y = b(x) \\ y'(x_0) = y'_0 \\ y(x_0) = y_0 \end{array}}$$

second order

In either case, if the $a_i(x)$ and $b(x)$ are all analytic at x_0, then there exists a power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

centered at x_0.

Furthermore, the radius of convergence $r > 0$ for the power series of the solution $y(x)$ is at least the smallest radius of convergence from amongst the power series of the $a_i(x)$ and $b(x)$

Ex: Suppose $a_0(x)$ is analytic at x_0 with radius of convergence is $r = 3$.
Suppose $b(x)$ is analytic at x_0 with radius of convergence $r = 800$.

Then,

$$\boxed{\begin{aligned} y' + a_0(x)\, y &= b(x) \\ y(x_0) &= y_0 \end{aligned}}$$

has a power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with radius of convergence at least $r = 3$.

minimum of 3 and 800 from above