Math 2150-02
11/10/25

HW 8 #1(c)

Solve

$$y'' + 4y' + 4y = 4x^2 - 8x$$

Step 1: Solve $y'' + 4y' + 4y = 0$

$$r^2 + 4r + 4 = 0$$

$$(r+2)(r+2) = 0$$

$$r = -2, -2$$

$$y_h = c_1 e^{-2x} + c_2 x e^{-2x}$$

Step 2:

Guess $y_p = Ax^2 + Bx + C$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Plug into $y'' + 4y' + 4y = 4x^2 - 8x$
to get:

$$\underbrace{(2A)}_{y_p''} + 4\underbrace{(2Ax + B)}_{y_p'} + 4\underbrace{(Ax^2 + Bx + c)}_{y_p} = 4x^2 - 8x$$

$$2A + 8Ax + 4B + 4Ax^2 + 4Bx + 4C = 4x^2 - 8x$$

$$\underbrace{4A}_{4}x^2 + \underbrace{(8A + 4B)}_{-8}x + \underbrace{(2A + 4B + 4C)}_{0} = 4x^2 - 8x$$

$$\boxed{\begin{aligned} 4A &= 4 \\ 8A + 4B &= -8 \\ 2A + 4B + 4C &= 0 \end{aligned}}$$

$$\boxed{A = 1}$$

$$\boxed{\begin{aligned} 8(1) + 4B &= -8 \\ B &= -4 \end{aligned}}$$

$$\boxed{\begin{aligned} 2(1) + 4(-4) + 4C &= 0 \\ 4C &= 14 \\ C &= \tfrac{14}{4} = \tfrac{7}{2} \end{aligned}}$$

$$y_p = Ax^2 + Bx + C = x^2 - 4x + \frac{7}{2}$$

Answer:

$$y = y_h + y_p$$
$$= c_1 e^{-2x} + c_2 x e^{-2x} + x^2 - 4x + \frac{7}{2}$$

HW 9 - #1(a)

Solve

$$y'' - 4y' + 4y = \underbrace{(x+1)e^{2x}}_{b(x)}$$

given that the general solution to

$$y'' - 4y' + 4y = 0 \quad \leftarrow \text{homogeneous eq.}$$

is $y_h = c_1 e^{2x} + c_2 x e^{2x}$

$$\boxed{y_1 = e^{2x}} \qquad \boxed{y_2 = xe^{2x}}$$

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$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 1\cdot e^{2x} + x(2e^{2x}) \end{vmatrix}$$

$$= (e^{2x})(e^{2x} + 2xe^{2x})$$

$$- (2e^{2x})(xe^{2x})$$

$$= e^{4x} + 2xe^{4x} - 2xe^{4x}$$

$$= e^{4x}$$

$$v_1 = \int \frac{-y_2\, b(x)}{W(y_1, y_2)}\, dx$$

$$= \int \frac{-x e^{2x} (x+1) e^{2x}}{e^{4x}}\, dx$$

$$= \int \frac{-x(x+1) \cancel{e^{4x}}}{\cancel{e^{4x}}}\, dx$$

$$= \int (-x^2 - x)\, dx = \boxed{-\frac{1}{3}x^3 - \frac{1}{2}x^2}$$

$$v_2 = \int \frac{y_1\, b(x)}{W(y_1, y_2)}\, dx$$

$$= \int \frac{\cancel{e^{2x}}(x+1)\cancel{e^{2x}}}{\cancel{e^{4x}}}\,dx \leftarrow \boxed{e^{2x}e^{2x} = e^{4x}}$$

$$= \int (x+1)\,dx$$

$$= \frac{1}{2}x^2 + x$$

Then,

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x}$$

And

$$y = y_h + y_p = c_1 e^{2x} + c_2 x e^{2x} + \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x}$$

## $\boxed{\text{HW 10 - \#1(a)}}$

Given that $y_1 = x^4$ solves

$$x^2 y'' - 7xy' + 16y = 0$$

on $I = (0, \infty)$, find the general solution.

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Divide by $x^2$ to get

$$y'' \underbrace{- \frac{7}{x}}_{a_1(x) = -\frac{7}{x}} y' + \frac{16}{x^2} y = 0$$

Then,

$$y_2 = y_1 \cdot \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx$$

$$= x^4 \int \frac{e^{-\int \left(-\frac{7}{x}\right) dx}}{(x^4)^2}\, dx$$

$$= x^4 \int \frac{e^{7\int \frac{1}{x} dx}}{x^8}\, dx$$

$$= x^4 \int \frac{e^{7\ln|x|}}{x^8}\, dx$$

$I = (0, \infty)$

$x > 0$

$|x| = x$

$$= x^4 \int \frac{e^{7\ln(x)}}{x^8}\, dx$$

$$A\ln(B) = \ln(B^A)$$

$$= x^4 \int \frac{e^{\ln(x^7)}}{x^8}\,dx$$

$$e^{\ln(A)} = A$$

$$= x^4 \int \frac{x^7}{x^8}\,dx$$

$$= x^4 \int \frac{1}{x}\,dx$$

$$= x^4 \ln|x|$$

$$x > 0$$
$$|x| = x$$

$$= x^4 \ln(x)$$

Answer: $y_h = c_1 y_1 + c_2 y_2$
$$= c_1 x^4 + c_2 x^4 \ln(x)$$

HW 11-1(a)

Find the power series for

$f(x) = x^3 + x$

centered at $x_0 = 1$.

$f(x) = x^3 + x \longrightarrow f(1) = 1^3 + 1 = 2$

$f'(x) = 3x^2 + 1 \longrightarrow f'(1) = 3(1)^2 + 1 = 4$

$f''(x) = 6x \longrightarrow f''(1) = 6(1) = 6$

$f'''(x) = 6 \longrightarrow f'''(1) = 6$

$f''''(x) = 0$ $\qquad$ $f''''(1) = 0$

$\downarrow$ all zero after $\qquad$ $\downarrow$ all zero after

$$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$$

$$= 2 + 4(x-1) + \frac{6}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3 + 0 + 0 + 0 + \cdots$$

$$= \boxed{2 + 4(x-1) + 3(x-1)^2 + (x-1)^3}$$

## HW 12

1. Find the first <u>four</u> non-zero terms of a power series solution to

$$\boxed{y'' - xy = 0}$$

$$\boxed{y'(0)=1,\ y(0)=2}$$

What's the radius of convergence of your answer?

We will center our power series at $x_0=0$.

$$y''-xy=0$$

coefficients:

$-x$

$0$

polynomials, they are analytic, $r=\infty$ each

our solution will have radius of convergence $r=\infty$

Find answer.

Given: $y(0) = 2$, $y'(0) = 1$

$y'' - xy = 0$

$y'' = xy$

$y''(0) = (0)[y(0)]$

$= (0)[2] = 0$

$y''(0) = 0$

$y'' = xy$

$y''' = 1 \cdot y + xy'$

$y'''(0) = y(0) + (0)[y'(0)]$

$$= 2 + (0)[1]$$

$$= 2$$

$$\boxed{y'''(0) = 2}$$

$$y''' = y + xy'$$

$$y'''' = y' + 1 \cdot y' + x \cdot y''$$

$$= 2y' + xy''$$

$$y''''(0) = 2[y'(0)] + (0)[y''(0)]$$

$$= 2[1] + (0)[0]$$

$$= 2$$

$$\boxed{y''''(0) = 2}$$

$$y(x) = y(0) + y'(0)(x-0) + \frac{y''(0)}{2!}(x-0)^2 + \frac{y'''(0)}{3!}(x-0)^3 + \frac{y''''(0)}{4!}(x-0)^4 + \cdots$$

$$= 2 + 1\cdot x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{2}{4!}x^4 + \cdots$$

$$= \boxed{2 + x + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \cdots}$$

radius of convergence $r = \infty$