Math 2150-02

10/6/25

$\boxed{\text{HW 3}}$

$\boxed{2(b)}$ Solve the linear ODE

$$x^2 y' + x(x+2)y = e^x$$

on $I = (0, \infty) \leftarrow \boxed{0 < x}$

Divide by x^2 to get

$$y' + \boxed{\frac{x(x+2)}{x^2}}\, y = \frac{1}{x^2} e^x$$

$$\boxed{\frac{x(x+2)}{x^2} = \frac{x+2}{x} = 1 + \frac{2}{x}}$$

So,

$$y' + \underbrace{\left(1 + \frac{2}{x}\right)}_{a(x)} y = \frac{1}{x^2} e^x$$

We have

$$A(x) = \int \left(1 + \frac{2}{x}\right) dx$$

$$= x + 2\ln|x|$$

$$= x + 2\ln(x)$$

$0 < x$

So,

$$e^{A(x)} = e^{x + 2\ln(x)} = e^x e^{2\ln(x)}$$

$e^{\ln(c)} = c$

$$= e^x e^{\ln(x^2)} = e^x \cdot x^2 = x^2 e^x$$

$A\ln(B) = \ln(B^A)$

Multiply

$$y' + \left(1 + \frac{2}{x}\right) y = \frac{1}{x^2} e^x$$

by $e^{A(x)} = x^2 e^x$ to get:

$$\underbrace{x^2 e^x y' + x^2 e^x \left(1 + \frac{2}{x}\right) y}_{\left(e^{A(x)} \cdot y\right)'} = \underbrace{x^2 e^x \left(\frac{1}{x^2} e^x\right)}_{e^{2x}}$$

So,

$$\left(x^2 e^x \cdot y\right)' = e^{2x}$$

Integrate both sides:

$$x^2 e^x y = \frac{1}{2} e^{2x} + C$$

So,

$$y = \frac{1}{x^2 e^x}\left(\frac{1}{2}e^{2x} + C\right)$$

$$\boxed{y = \frac{1}{2}\,\frac{1}{x^2}\,e^x + C\,\frac{1}{x^2 e^x}}$$

HW 4

① (c) Solve the separable IVP

$$\underbrace{\frac{dy}{dx} = \frac{-x}{y}}_{\text{ODE}}, \quad \underbrace{y(4) = 3}_{\text{when } x=4,\ y=3}$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

$$\boxed{\frac{1}{2}y^2 = -\frac{x^2}{2} + C}$$

$$\frac{1}{2}y^2 + C_1 = \frac{-x^2}{2} + C_2$$

$$\frac{1}{2}y^2 = \frac{-x^2}{2} + \underbrace{(C_2 - C_1)}_{C}$$

Let's find C.

We want $y(4) = 3$.

Plug in $x = 4, y = 3$ to get!

$$\frac{1}{2}(3)^2 = -\frac{(4)^2}{2} + C$$

$$\frac{9}{2} = \frac{-16}{2} + C$$

$$25/2 = C$$

So, $\frac{1}{2}y^2 = -\frac{x^2}{2} + \frac{25}{2}$

Thus,

$$y^2 = 25 - x^2$$

$$y = \pm\sqrt{25 - x^2}$$

Is it + or − ?

We want $y(4) = 3$.

Need $3 = \pm\sqrt{25 - (4)^2}$

Need $3 = \pm\sqrt{9}$

Need +.

So, $\boxed{y = \sqrt{25 - x^2}}$

$\boxed{\text{HW 5}}$

① (d) $\dfrac{2x}{y} - \dfrac{x^2}{y^2} y' = 0$

Test for exactness:

$M = \dfrac{2x}{y} = 2xy^{-1}$

$N = \dfrac{-x^2}{y^2} = -x^2y^{-2}$

$\dfrac{\partial M}{\partial y} = 2x(-y^{-2}) = -2xy^{-2}$

$\dfrac{\partial N}{\partial x} = -(2x)y^{-2} = -2xy^{-2}$

EQUAL

So, the equation is exact.

Now we solve it

Need f where

$$\boxed{\begin{aligned}\frac{\partial f}{\partial x} &= M \\ \frac{\partial f}{\partial y} &= N\end{aligned}} \rightarrow \boxed{\begin{aligned}\frac{\partial f}{\partial x} &= 2xy^{-1} \quad (1) \\ \frac{\partial f}{\partial y} &= -x^2y^{-2} \quad (2)\end{aligned}}$$

Integrate (1) with respect to x:

$$f(x,y) = x^2y^{-1} + \underbrace{C(y)}_{\text{constant with respect to } x}$$

Integrate (2) with respect to y:

$$f(x,y) = -x^2\frac{y^{-1}}{-1} + \underbrace{D(x)}_{\text{constant with respect to } y}$$

$$= x^2y^{-1} + D(x)$$

Set equal to get:

$$\cancel{x^2y^{-1}} + C(y) = \cancel{x^2y^{-1}} + D(x)$$

$$C(y) = D(x)$$

Set $C(y) = 0$ and $D(x) = 0$.

So, $f(x,y) = x^2y^{-1} + D(x)$

$$= x^2y^{-1} + 0$$

$$= x^2y^{-1}.$$

Answer: $x^2y^{-1} = c$

$$\boxed{y = \frac{1}{c} x^2}$$

HW 6

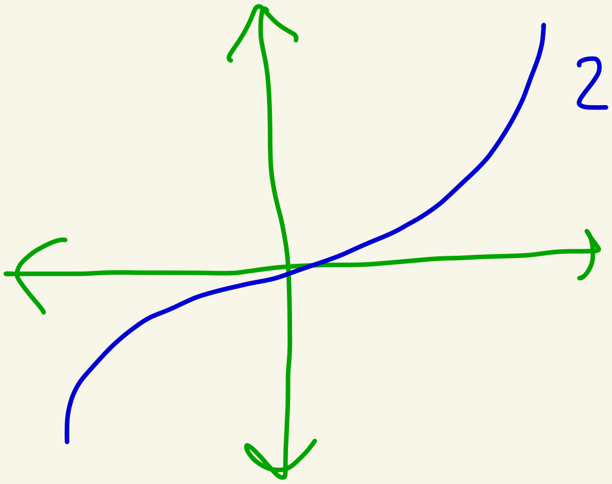
② (a) Show that $y_1 = x^2$, $y_2 = x^4$ are linearly independent on $I = (-\infty, \infty)$.

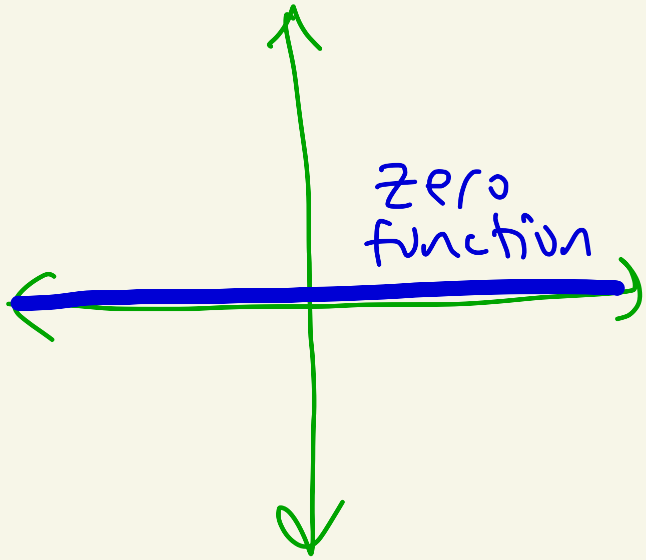
$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^4 \\ 2x & 4x^3 \end{vmatrix}$$

$$= (x^2)(4x^3) - (x^4)(2x)$$

$$= 2x^5$$

Is this the zero function?





No, for example at $x=1$ the Wronskian is $2(1)^5=2$ which is not 0.

So, $y_1=x^2$ and $y_2=x^4$ are linearly independent.

2(b) Show that $y_1=x^2$ and $y_2=x^4$ both solve

$$x^2y'' - 5xy' + 8y = 0$$

$$\boxed{\begin{aligned} y_1 &= x^2 \\ y_1' &= 2x \\ y_1'' &= 2 \end{aligned}}$$

Plug in $y_1 = x^2$ first:

$$x^2\underbrace{(2)}_{y_1''} - 5x\underbrace{(2x)}_{y_1'} + 8\underbrace{(x^2)}_{y_1}$$

$$= 2x^2 - 10x^2 + 8x^2 = 0$$

So, y_1 solves the equation.

$$\boxed{\begin{aligned} y_2 &= x^4 \\ y_2' &= 4x^3 \\ y_2'' &= 12x^2 \end{aligned}}$$

Plug in $y_2 = x^4$ next:

$$x^2\underbrace{(12x^2)}_{y_2''} - 5x\underbrace{(4x^3)}_{y_2'} + 8\underbrace{(x^4)}_{y_2}$$

$$= 12x^4 - 20x^4 + 8x^4 = 0$$

So, y_2 solves the equation

$\boxed{2(c)}$ What is the general solution to

$$x^2 y'' - 5xy' + 8y = 0 \;?$$

Answer:

$$\boxed{y_h = c_1 x^2 + c_2 x^4}$$

$$\boxed{c_1 y_1 + c_2 y_2}$$

$\boxed{\text{HW 7}}$

1 (e) Solve $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 16y = 0.$

$$y'' + 8y' + 16y = 0$$

$$r^2 + 8r + 16 = 0$$

$$\underbrace{(r+4)}_{\substack{r+4=0 \\ r=-4}}\underbrace{(r+4)}_{\substack{r+4=0 \\ r=-4}} = 0$$

There is a repeated real root $r = -4$.

Answer: $\boxed{y_h = c_1 e^{-4x} + c_2 x e^{-4x}}$