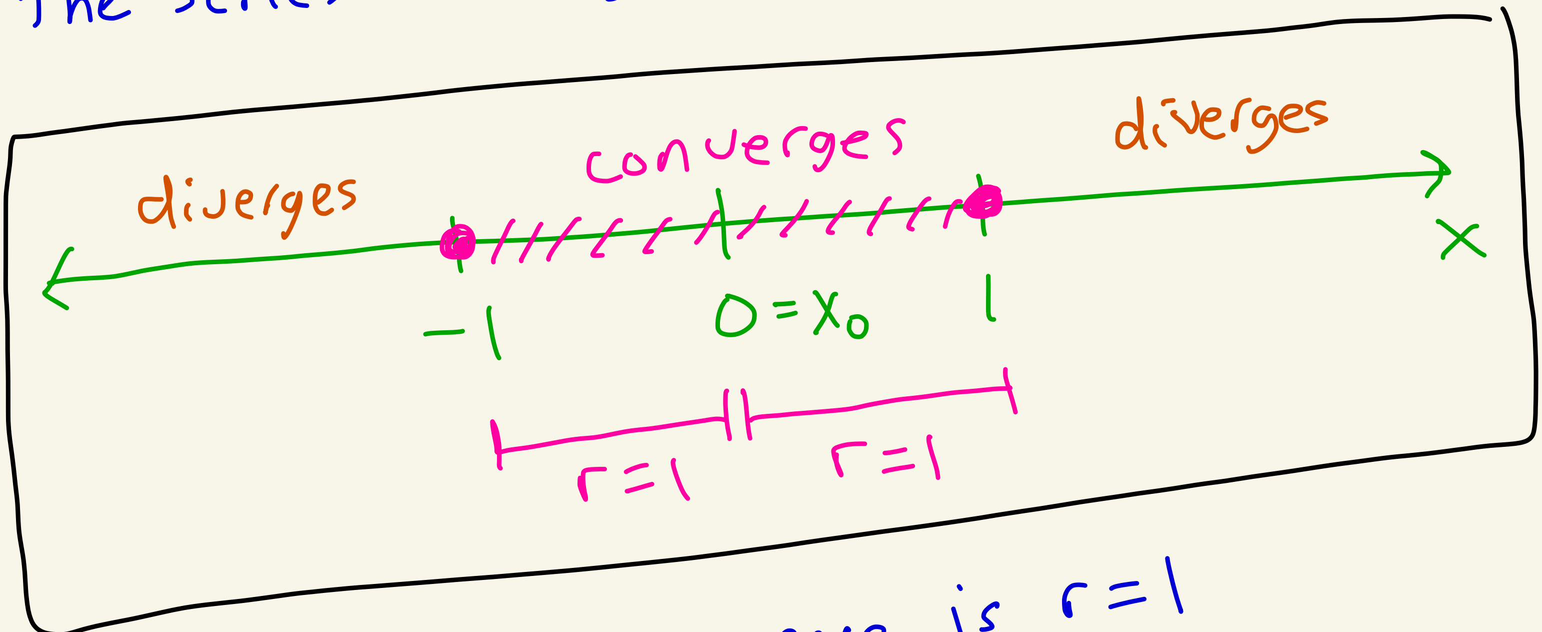
Math 2150-02
10/29/25

Ex: From Calc II we have

$$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \cdots$$

centered at $x_0 = 0$

the series converges when $-1 \leq x \leq 1$



the radius of convergence is $r = 1$

Side comment:

You can use this to approximate π.

Plug $x=1$ in to get:

$$\frac{\pi}{4} = \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right]$$

Taylor formula

If

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

has radius of convergence $r > 0$,

then

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

$f^{(n)}$ is the nth derivative

So,

$$f(x) = \underbrace{\frac{f^{(0)}(x_0)}{0!}}_{a_0} + \underbrace{\frac{f'(x_0)}{1!}}_{a_1}(x-x_0)^1 + \underbrace{\frac{f''(x_0)}{2!}}_{a_2}(x-x_0)^2 + \cdots$$

$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

Ex: Find the power series for $f(x) = x^2$ centered at $x_0 = 2$

Note: $f(x) = x^2$ is already written as a power series, but centered at $x_0 = 0$.

$$x^2 = 0 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4 + \cdots$$

We will recenter it around $x_0 = 2$

$f(x) = x^2 \longrightarrow f(2) = 2^2 = 4$

$f'(x) = 2x \longrightarrow f'(2) = 2(2) = 4$

$f''(x) = 2 \longrightarrow f''(2) = 2$

$f'''(x) = 0$

$f^{(4)}(x) = 0$

$\vdots \quad \vdots$

$f^{(n)}(x) = 0$, $n \geq 3$ $\longrightarrow f^{(n)}(2) = 0$, $n \geq 3$

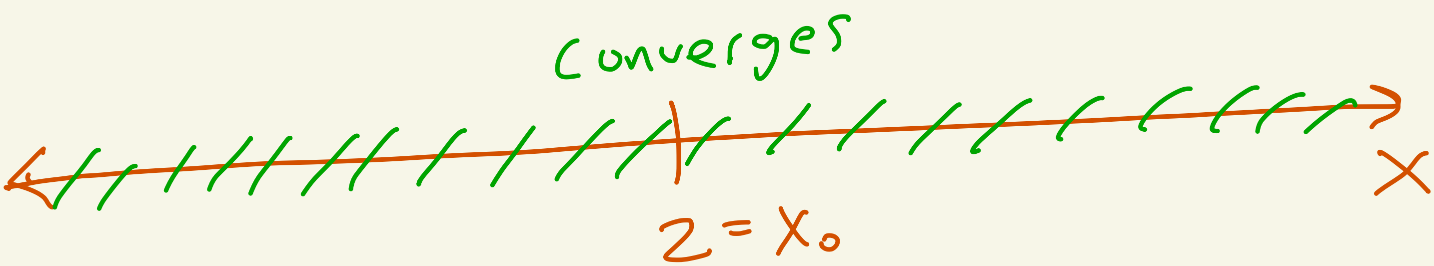
$$x^2 = f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^{(4)}(2)}{4!}(x-2)^4 + \cdots$$

$$= 4 + 4(x-2) + \frac{2}{2!}(x-2)^2 + \underbrace{\frac{0}{3!}(x-2)^3 + \frac{0}{4!}(x-2)^4 + \cdots}_{0}$$

We get

$$\boxed{x^2 = 4 + 4(x-2) + (x-2)^2}$$

This series converges for all x.
The radius of convergence is $r = \infty$

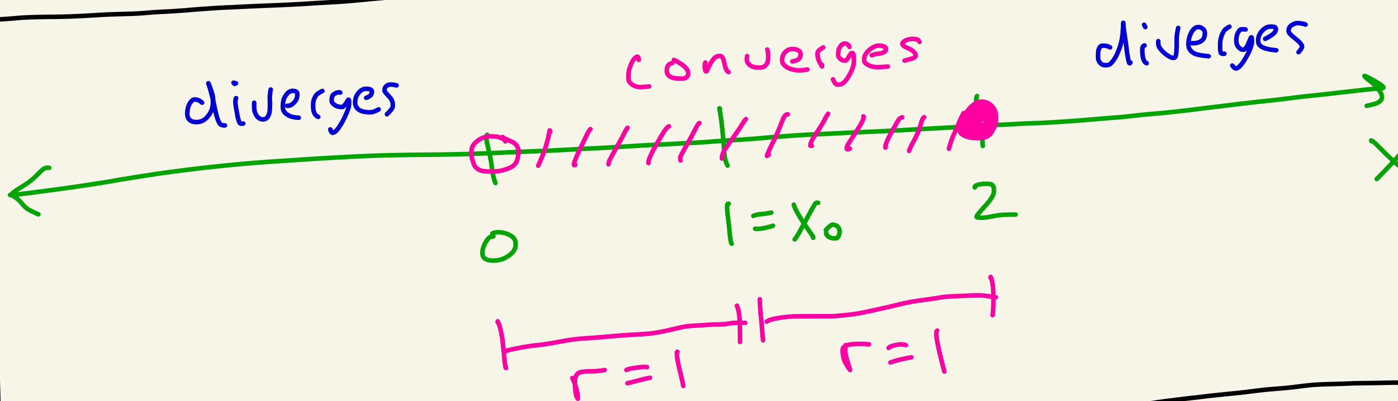


Fact: You can differentiate or integrate a function by differentiating or integrating its power/taylor series term by term. When you do this, the result has the same radius of convergence as the original function.

Ex: Last time we recalled that

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

when $0 < x \leq 2$.



Differentiate both sides to get:

$$\frac{1}{x} = 1 - \frac{1}{2}\left[2(x-1)^1\right] + \frac{1}{3}\left[3(x-1)^2\right] - \frac{1}{4}\left[4(x-1)^3\right] + \cdots$$

So,

$$\frac{1}{x} = 1-(x-1)+(x-1)^2-(x-1)^3+\cdots$$

This series will also have radius of convergence $r=1$. But the endpoint convergence changes. It will converge for $0<x<2$.