Math 2150-02
10/27/25

(Topic 11 continued...)

Def: A <u>power series</u> is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \underbrace{a_0}_{\substack{n=0\\ \text{term}}} + \underbrace{a_1(x-x_0)}_{n=1 \text{ term}} + \underbrace{a_2(x-x_0)^2}_{n=2 \text{ term}} + \underbrace{a_3(x-x_0)^3}_{n=3 \text{ term}} + \cdots$$

Where the $a_0, a_1, a_2, \ldots$ are constants, x_0 is a constant (called the <u>center</u> of the series), and x is a variable.

Ex: (Geometric series)

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$\boxed{x_0}$

$$= \underbrace{1}_{a_0} + \underbrace{1}_{a_1} \cdot (x-0) + \underbrace{1}_{a_2} \cdot (x-0)^2$$

$$+ \underbrace{1}_{a_3} \cdot (x-0)^3 + \underbrace{1}_{a_4} \cdot (x-0)^4 + \cdots$$

$\boxed{x_0}$

the center is $x_0 = 0$

In Calc II, you learn that the above series converges when $-1 < x < 1$ and diverges otherwise.

And when $-1<x<1$ we get

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

$\boxed{-1<x<1}$

For example if $x = \frac{9}{10} = 0.9$, then

$$1 + \left(\frac{9}{10}\right) + \left(\frac{9}{10}\right)^2 + \left(\frac{9}{10}\right)^3 + \cdots = \frac{1}{1-\frac{9}{10}} = 10$$

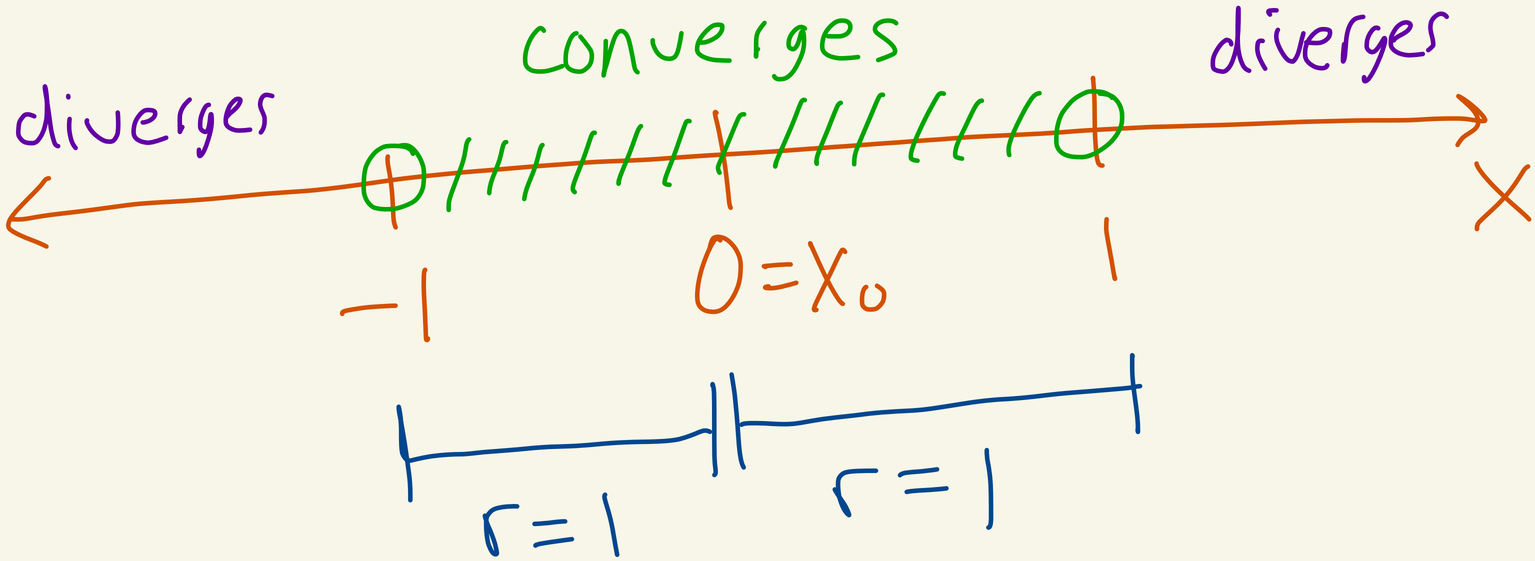
$\boxed{-1 < \frac{9}{10} < 1}$

If $x = 1$, then

$$1 + 1 + 1^2 + 1^3 + 1^4 + 1^5 + \cdots$$

diverges.

Here's a picture of what x's make $\sum_{n=0}^{\infty} x^n$ converge/diverge.



$r=1$ is called the radius of convergence of $\sum_{n=0}^{\infty} x^n$

Ex: In Calc II you learn that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

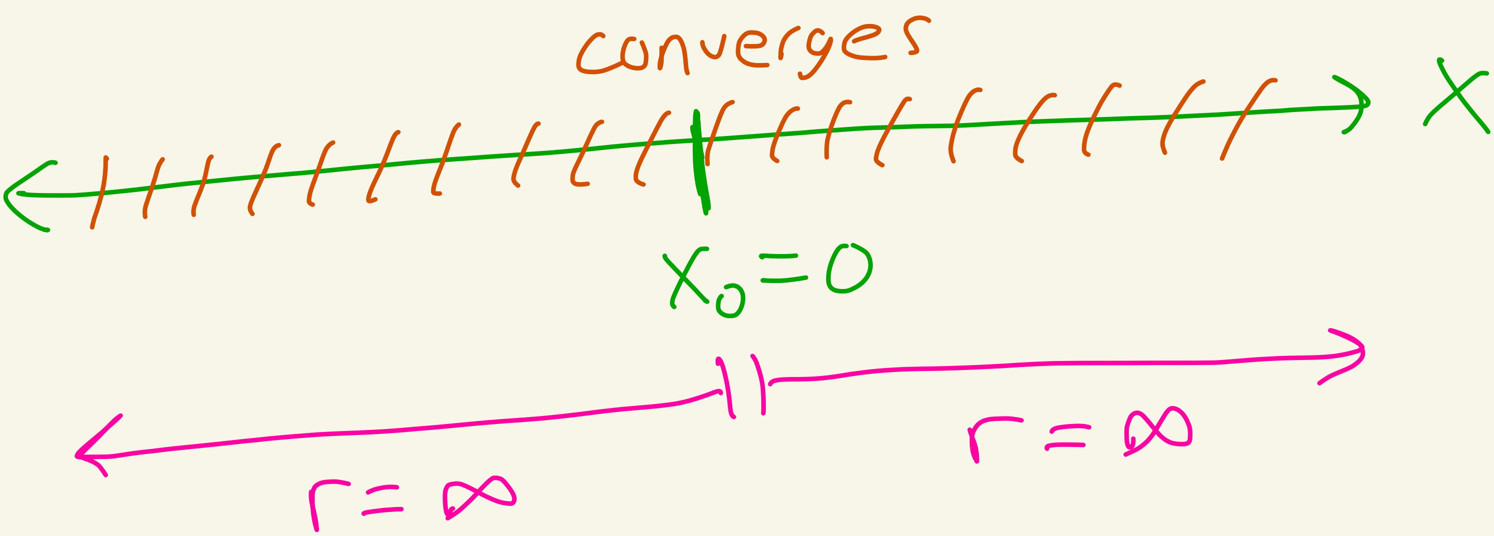
$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$+ \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$\boxed{\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 2\cdot 1 = 2 \\ 3! &= 3\cdot 2\cdot 1 = 6 \\ 4! &= 4\cdot 3\cdot 2\cdot 1 \\ &= 24 \end{aligned}}$$

The center is $x_0 = 0$

This series converges for all x.



The radius of convergence is $r = \infty$

For example, when $x=1$ we get

$$e^1 = 1+1+\frac{1^2}{2!}+\frac{1^3}{3!}+\frac{1^4}{4!}+\frac{1^5}{5!}+\cdots$$

$$e = 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots$$

<u>Ex:</u> In Calc II you learn that

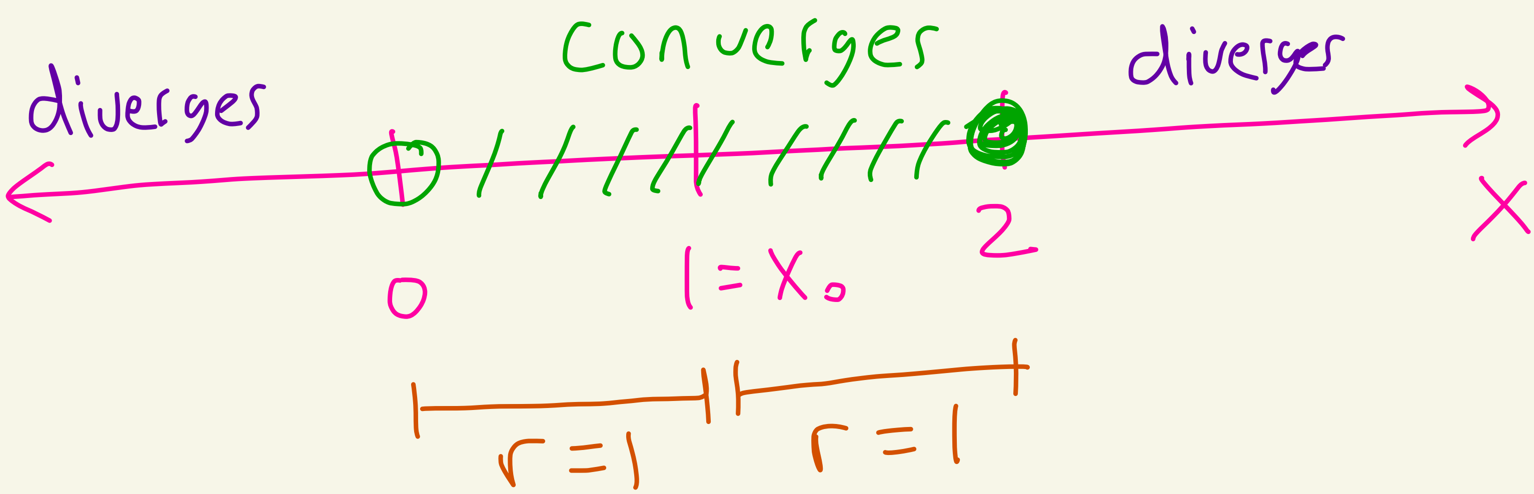
$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

$$\boxed{0<x\leq 2}$$

$$= \underbrace{(x-1)}_{\substack{n=1\\ \text{term}}} - \underbrace{\frac{1}{2}(x-1)^2}_{\substack{n=2\\ \text{term}}} + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

The center is $x_0 = 1$.

picture of convergence/divergence:



For example,

$$\ln\left(\frac{1}{2}\right) = \left(\frac{1}{2}-1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)^2 + \frac{1}{3}\left(\frac{1}{2}-1\right)^3 - \frac{1}{4}\left(\frac{1}{2}-1\right)^4 + \cdots$$

$\boxed{x = 1/2}$

$\boxed{0 < \frac{1}{2} \leq 2}$

$$= \frac{1}{2} - \frac{1}{2}\left(-\frac{1}{2}\right)^2 + \frac{1}{3}\left(-\frac{1}{2}\right)^3 - \frac{1}{4}\left(-\frac{1}{2}\right)^4 + \cdots$$

In general, consider a power series centered at x_0:

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

There are 3 possibilities:

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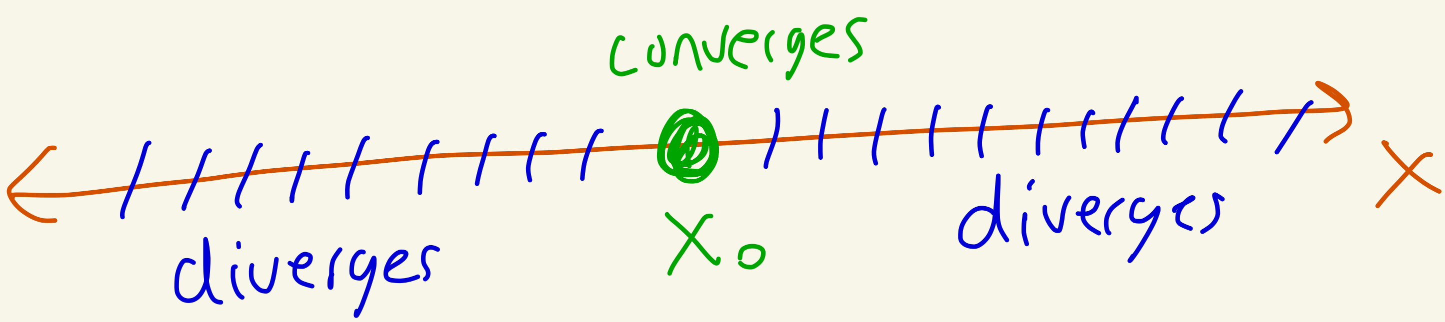
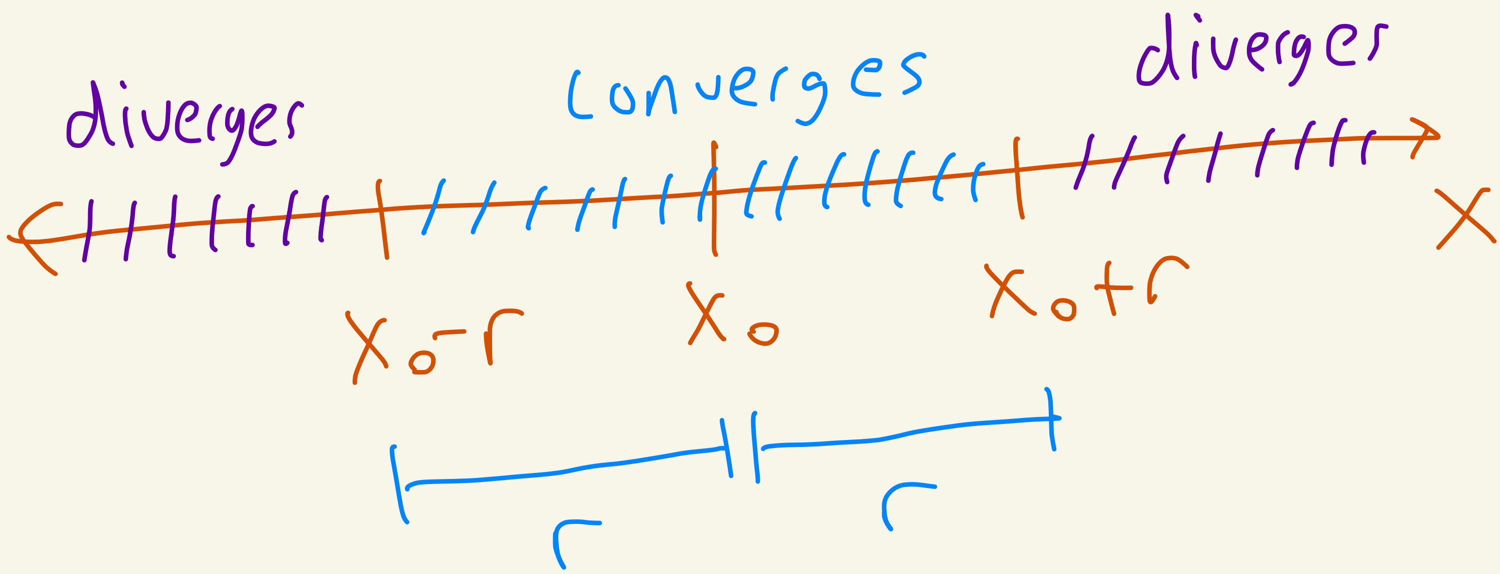
① The series converges only at $x = x_0$



Here the radius of convergence is $r = 0$.

(2) there exists a radius of convergence $r > 0$ where the series converges when

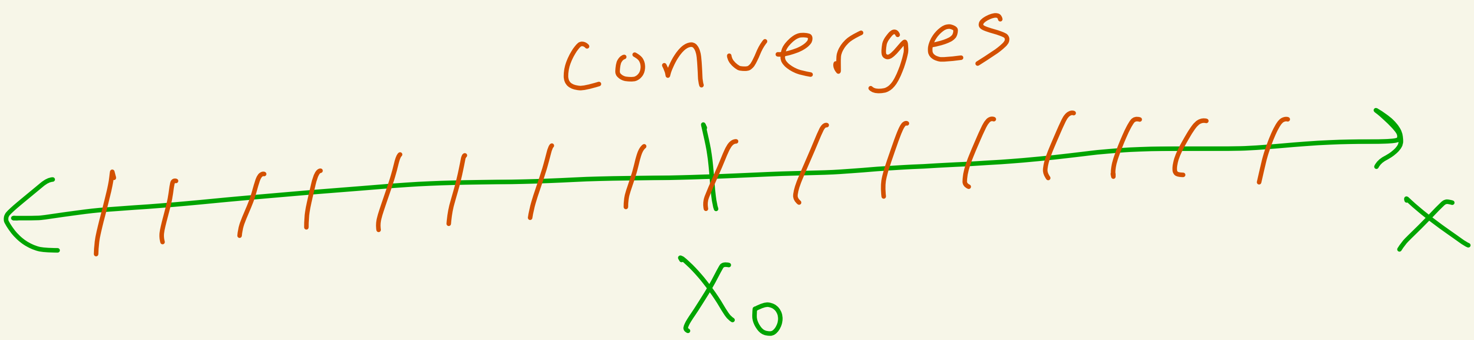
$$x_0 - r < x < x_0 + r$$



It diverges when $x < x_0 - r$ and when $x_0 + r < x$.

It may converge or diverge at the endpoints $x = x_0 - r$ and $x = x_0 + r$

③ The radius of convergence is $r = \infty$ and the series converges for all x.



Ex: From Calc II:

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

These both have radius of convergence $r = \infty$

For example,

$$\sin(2) = 2 - \frac{1}{3!}(2^3) + \frac{1}{5!}(2^5) - \frac{1}{7!}(2^7)$$
$$+ \frac{1}{9!}(2^9) - \frac{1}{11!}(2^{11})$$
$$+ \frac{1}{13!}(2^{13}) - \cdots$$