Math 2150-02

10/22/25

Topic 10 – Reduction of order

Suppose you know one solution y_1 to the homogeneous ODE

$$\boxed{y'' + a_1(x)y' + a_0(x)y = 0} \quad (*)$$

on an interval I where $y_1(x) \neq 0$ on I. Then one can find another solution using

$$y_2 = y_1 \left[\int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx \right]$$

Furthermore, y_1 and y_2 will be linearly independent on I. Thus,

$$y_h = c_1 y_1 + c_2 y_2$$

will be the general solution to $(*)$ on I.

Ex: Consider

$$(x^2+1)y'' - 2xy' + 2y = 0$$

on $I = (0, \infty)$

Let $y_1 = x$.

Note $y_1 \neq 0$ on I since $x > 0$ on I

Let's check if $y_1 = x$ solves the equation.

Plug it in to get:

$$(x^2+1)\underbrace{(0)}_{y_1''} - 2x\underbrace{(1)}_{y_1'} + 2\underbrace{(x)}_{y_1} = 0$$

So, y_1 solves the equation.

Let's use our formula to find a second solution.

First divide the equation by (x^2+1) to get:

$$\underbrace{y''}_{1 \cdot y''} - \underbrace{\frac{2x}{(x^2+1)}}_{a_1(x) = \frac{-2x}{x^2+1}} y' + \frac{2}{(x^2+1)} y = 0$$

$\boxed{1 \cdot y''}$ ✓

$a_1(x) = \frac{-2x}{x^2+1}$

We get

$$y_2 = y_1 \left[\int \frac{e^{-\int a_1(x)\,dx}}{y_1^2} \, dx \right]$$

$$= x \int \frac{e^{-\int \left(\frac{-2x}{x^2+1}\right) dx}}{x^2} \, dx$$

$$= x \int \frac{e^{\int \frac{2x}{x^2+1}dx}}{x^2} dx$$

$$\int \frac{2x}{x^2+1} dx = \int \frac{1}{u} du$$

$$u = x^2+1$$
$$du = 2x\,dx$$

$$= \ln|u| = \ln|x^2+1| = \ln(x^2+1)$$

$$x^2+1 > 0$$

$$= x \int \frac{e^{\ln(x^2+1)}}{x^2} dx$$

$$= x \int \frac{x^2+1}{x^2}\, dx$$

$$\boxed{e^{\ln(z)} = z}$$

$$= x \int \left(\frac{x^2}{x^2} + \frac{1}{x^2} \right) dx$$

$$= x \int \left(1 + x^{-2} \right) dx$$

$$= x \left[x + \frac{x^{-1}}{-1} \right]$$

$$= x \left(x - \frac{1}{x} \right)$$

$$= x^2 - 1$$

Thus, the general solution to

$$(x^2+1)y'' - 2xy' + 2y = 0$$

on $I = (0,\infty)$ is

$$y_h = c_1 y_1 + c_2 y_2$$
$$= c_1 x + c_2 (x^2 - 1)$$

<u>Ex:</u> Given that $y_1 = x^4$ is a solution to

$$x^2 y'' - 7xy' + 16y = 0$$

on $I = (0,\infty)$, find the general solution.

Divide by x^2 to get:

$$y'' - \frac{7}{x} y' + \frac{16}{x^2} y = 0$$

$\boxed{1}$ ✓

$\boxed{a_1(x) = \frac{-7}{x}}$

We get:

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx$$

$$= x^4 \int \frac{e^{-\int \left(-\frac{7}{x}\right) dx}}{\left(x^4\right)^2}\,dx$$

$$= x^4 \int \frac{e^{7\int \frac{1}{x} dx}}{x^8} dx$$

$$= x^4 \int \frac{e^{7\ln|x|}}{x^8} dx$$

$I = (0, \infty)$
$x > 0$
$|x| = x$

$$= x^4 \int \frac{e^{7\ln(x)}}{x^8} dx$$

$$= x^4 \int \frac{e^{\ln(x^7)}}{x^8} dx$$

$c\ln(z)$
$= \ln(z^c)$

$$= x^4 \int \frac{x^7}{x^8}\,dx$$

$e^{\ln(z)} = z$

$$= x^4 \int \frac{1}{x}\,dx$$

$$= x^4 \ln|x|$$

$I = (0, \infty)$

$x > 0$

$|x| = x$

$$= x^4 \ln(x)$$

Summary: The general solution to

$$x^2 y'' - 7xy' + 16y = 0$$

on $I = (0, \infty)$ is

$$y_h = c_1 x^4 + c_2 x^4 \ln(x)$$

Topic 11 - Review of power series

Def: An <u>infinite sum/series</u> is a sum of the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots$$

we started at $n=0$, but it doesn't have to.

To define what this means we need partial sums.

Let

$$S_N = a_0 + a_1 + a_2 + \cdots + a_N$$

So,

$$S_0 = a_0$$

$$S_1 = a_0 + a_1$$
$$S_2 = a_0 + a_1 + a_2$$
$$S_3 = a_0 + a_1 + a_2 + a_3$$
$$\vdots \qquad \vdots$$

and so on.

If $\lim\limits_{N\to\infty} S_N$ exists and equals L, then we say that $\sum\limits_{n=0}^{\infty} a_n$ converges and write $\sum\limits_{n=0}^{\infty} a_n = L$.

If $\lim\limits_{N\to\infty} S_N$ does not exist then we say $\sum\limits_{n=0}^{\infty} a_n$ diverges.

Ex: Consider

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

Let's make a table of partial sums.

N	S_N
0	1
1	$1+\frac{1}{2} = 1.5$
2	$1+\frac{1}{2}+\frac{1}{2^2} = 1.75$
3	$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3} = 1.875$
4	$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4} = 1.9375$

5	1.96875
⋮	⋮
50	$1.\underbrace{999\ldots99}_{15\ 9\text{'s}}11182\ldots$
⋮	
100	$1.\underbrace{999\ldots99}_{30\ 9\text{'s}}11139\ldots$

From Calculus you get

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2$$

$r = \frac{1}{2}$

It comes from:

If $-1 < r < 1$,

then $$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Geometric sum