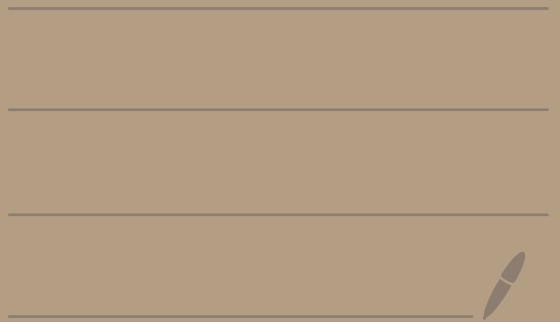


Math 2150-02
10/20/25



Ex: Solve

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Step 1: Solve $y'' - 4y' + 4y = 0$

The characteristic poly is

$$r^2 - 4r + 4 = 0$$

$$(r-2)(r-2) = 0$$

$$r = 2, 2$$

So,

$$y_h = c_1 \boxed{e^{2x}} + c_2 \boxed{x e^{2x}}$$

$\uparrow$ $\boxed{y_1}$ $\uparrow$ $\boxed{y_2}$

Step 2: Find y_p for

$b(x)$

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Use: $y_p = V_1 y_1 + V_2 y_2$ where

$$V_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)} dx, \quad V_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)} dx$$

Let's calculate $W(y_1, y_2)$.

We have

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & e^{2x} + 2x e^{2x} \end{vmatrix} \end{aligned}$$

$$= (e^{2x})(e^{2x} + 2xe^{2x}) - (2e^{2x})(xe^{2x})$$

$$= e^{4x} + 2xe^{4x} - 2xe^{4x}$$

$$= e^{4x}$$

So,

$$V_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)} dx$$

$$= \int \frac{-(xe^{2x})(x+1)e^{2x}}{e^{4x}} dx$$

$$= \int \frac{-x(x+1)\cancel{e^{2x}}\cancel{e^{2x}}}{\cancel{e^{4x}}} dx \quad \left[\begin{matrix} e^{2x} e^{2x} \\ \searrow \\ e^{4x} \end{matrix} \right]$$

$$= \int -x(x+1) dx$$

$$= \int (-x^2 - x) dx$$

$$= \boxed{-\frac{1}{3}x^3 - \frac{1}{2}x^2}$$

V_1

And,

$$V_2 = \int \frac{y_1 b(x)}{w(y_1, y_2)} dx$$

$$= \int \frac{e^{2x} (x+1) e^{2x}}{e^{4x}} dx$$

$$= \int \frac{\cancel{e^{4x}} (x+1)}{\cancel{e^{4x}}} dx$$

$$= \int (x+1) dx$$

$$= \boxed{\frac{1}{2}x^2 + x}$$

v_2

Thus,

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x}$$

Step 3: The general solution to

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{2x} + c_2 x e^{2x} \\ &\quad + \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right) e^{2x} \\ &\quad + \left(\frac{1}{2}x^2 + x\right) x e^{2x} \end{aligned}$$

Ex: Solve $y'' + y = \tan(x)$

Step 1: Solve $y'' + y = 0$.

The characteristic poly. is

$$r^2 + 1 = 0$$

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{\pm \sqrt{-4}}{2} = \frac{\pm \sqrt{4} \sqrt{-1}}{2}$$

$$= \frac{\pm 2i}{2} = \pm i$$

$$= \underbrace{0 \pm 1 \cdot i}_{\alpha \pm \beta i}$$

$$\alpha = 0, \beta = 1$$

Then,

$$y_h = c_1 e^{0x} \cos(1 \cdot x) + c_2 e^{0x} \sin(1 \cdot x)$$

$$c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

$$= c_1 \boxed{\cos(x)} + c_2 \boxed{\sin(x)}$$

$e^0 = 1$ (circled in pink) with an arrow pointing to the c_1 term.

y_1 (boxed in orange) with an arrow pointing to $\cos(x)$.

y_2 (boxed in orange) with an arrow pointing to $\sin(x)$.

Step 2: Find y_p for

$$y'' + y = \boxed{\tan(x)}$$

$b(x)$ (boxed in pink) with an arrow pointing to $\tan(x)$.

Let's calculate $W(y_1, y_2)$.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

$$= (\cos(x))(\cos(x)) - (-\sin(x))(\sin(x))$$

$$= \cos^2(x) + \sin^2(x)$$

$$= 1$$

We have

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)} dx$$

$$= \int \frac{-\sin(x) \tan(x)}{1} dx$$

$$= - \int \sin(x) \cdot \frac{\sin(x)}{\cos(x)} dx$$

$$= - \int \frac{\sin^2(x)}{\cos(x)} dx$$

$$= - \int \frac{1 - \cos^2(x)}{\cos(x)} dx$$

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 \\ \sin^2(x) &= 1 - \cos^2(x) \end{aligned}$$

$$= - \int \frac{1}{\cos(x)} dx + \int \frac{\cos^2(x)}{\cos(x)} dx$$

$$= -\int \sec(x) dx + \int \cos(x) dx$$

$$= \boxed{-\ln |\sec(x) + \tan(x)| + \sin(x)}$$


 V_1

And

$$V_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)} dx$$

$$= \int \frac{\cos(x) \cdot \tan(x)}{1} dx$$

$$= \int \cancel{\cos(x)} \cdot \frac{\sin(x)}{\cancel{\cos(x)}} dx$$

$$= \int \sin(x) dx$$

$$= -\cos(x)$$

V_2

Thus,

$$y_p = V_1 y_1 + V_2 y_2$$

$$= \left[-\ln|\sec(x) + \tan(x)| + \sin(x) \right] \cos(x) \\ + \left[-\cos(x) \right] \sin(x)$$

$$= -\cos(x) \cdot \ln|\sec(x) + \tan(x)|$$

Step 3: The general solution

to $y'' + y = \tan(x)$ is

$$y = y_h + y_p$$

$$= c_1 \cos(x) + c_2 \sin(x)$$

$$- \cos(x) \cdot \ln |\sec(x) + \tan(x)|$$
