Math 2150-02

10/15/26

Topic 9 - Variation of parameters

In topic 8 we learned a method to guess y_p using a table.

In this topic we find a formula for y_p using integrals.

It will work in situations that topic 8 doesn't work such as

$$y'' + y = \tan(x)$$

or when one doesn't have constant coefficients such as

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

Derivation time!

Suppose you already have two linearly independent solutions y_1 and y_2 to the homogeneous equation:

$$\boxed{y'' + a_1(x)y' + a_0(x)y = 0} \quad (*)$$

So the general solution to $(*)$ is $y_h = c_1y_1 + c_2y_2$.
We will use these to build a particular solution y_p to

$$\boxed{y'' + a_1(x)y' + a_0(x)y = b(x)} \quad (**)$$

To do this, set

$$y_p = v_1 y_1 + v_2 y_2$$

where v_1 and v_2 are unknown functions to be determined.

We will now plug y_p into (**) and find two conditions we must solve to ensure that y_p solves (**).

We have

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_p' = (v_1' y_1 + v_1 y_1') + (v_2' y_2 + v_2 y_2')$$

$$= (v_1 y_1' + v_2 y_2') + \underbrace{(v_1' y_1 + v_2' y_2)}_{\text{make condition that this is } 0}$$

We will simplify by making

the condition that

$$\boxed{v_1' y_1 + v_2' y_2 = 0} \leftarrow \boxed{\text{condition } 1}$$

With condition 1 satisfied we have:

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_p' = v_1 y_1' + v_2 y_2'$$

$$y_p'' = (v_1' y_1' + v_1 y_1'') + (v_2' y_2' + v_2 y_2'')$$

Plug this all into (**):

$$\boxed{y'' + a_1(x) y' + a_0(x) y = b(x)}$$

to get:

$$(v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2'') \leftarrow y_p''$$

$$+ a_1(x)[v_1 y_1' + v_2 y_2'] \leftarrow + a_1(x) y_p'$$

$$+ a_0(x)\left[v_1 y_1 + v_2 y_2\right] = b(x)$$

$$+ a_0(x) y_p = b(x)$$

This becomes:

$$v_1\big(\underbrace{y_1'' + a_1(x) y_1' + a_0(x) y_1}_{0}\big)$$
$$+ v_2\big(\underbrace{y_2'' + a_1(x) y_2' + a_0(x) y_2}_{0}\big)$$
$$+ \left(v_1' y_1' + v_2' y_2'\right) = b(x)$$

We get 0's above because y_1 and y_2 solve

$y'' + a_1(x) y' + a_0(x) y = 0$.

The above simplifies to

$$\boxed{v_1' y_1' + v_2' y_2' = b(x)} \leftarrow \boxed{\text{condition } 2}$$

Thus, $y_p = v_1 y_1 + v_2 y_2$ is a solution to

$$y'' + a_1(x) y' + a_0(x) y = b(x)$$

if

$$\boxed{\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ v_1' y_1' + v_2' y_2' &= b(x) \end{aligned}} \quad \begin{matrix} 1 \\ 2 \end{matrix}$$

The unknowns to solve for are v_1', v_2'.

Let's solve for v_2'.

Calculating

$$y_1' * 1 - y_1 * 2$$

to get:

$$\cancel{y_1' v_1' y_1} + y_1' v_2' y_2 = 0$$

$$-\left[\cancel{y_1 v_1' y_1'} + y_1 v_2' y_2' = y_1 \cdot b(x)\right]$$

$$y_1' v_2' y_2 - y_1 v_2' y_2' = -y_1 b(x)$$

So,

$$v_2'(y_1' y_2 - y_1 y_2') = -y_1 b(x)$$

Multiply by -1 to get

$$v_2'\big(\underbrace{y_1 y_2' - y_1' y_2}_{W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}\big) = y_1 b(x)$$

Thus,

$$v_2' = \frac{y_1 b(x)}{W(y_1, y_2)}$$

Hence,

$$v_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)} dx$$

If you calculate

$$y_2' * ① - y_2 * ②$$

you will get

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)} dx$$

Summary:

Suppose you have two linearly independent solutions y_1 and y_2 to the homogeneous equation

$$\boxed{y'' + a_1(x)y' + a_0(x)y = 0}$$

Then a particular solution y_p to

$$\boxed{y'' + a_1(x)y' + a_0(x)y = b(x)}$$

is

$$y_p = V_1 y_1 + V_2 y_2$$

where

$$V_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)}\,dx, \quad V_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)}\,dx$$