Math 2150-01
9/8/25

Ex: Solve $\frac{dy}{dx} = \frac{-x}{y}$

subject to $y(4) = 3$

We have

$$\frac{dy}{dx} = \frac{-x}{y}$$

So,

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

Use $y(4) = 3$ to find C,

Plug in $x=4$, $y=3$ to get:

$$\frac{3^2}{2} = -\left(\frac{4^2}{2}\right) + C$$

$$\frac{9}{2} = -8 + C$$

$$\frac{25}{2} = C$$

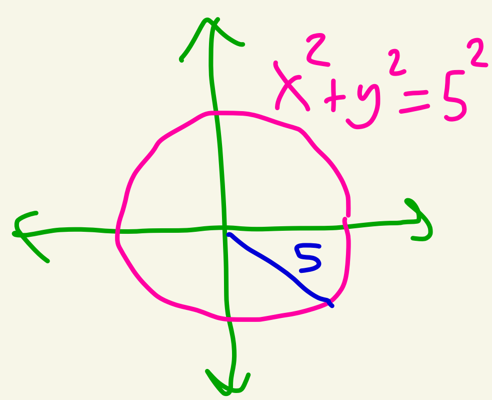
Thus,

$$\frac{y^2}{2} = \frac{-x^2}{2} + \frac{25}{2}$$

$C = {}^{25}/_{2}$

So,

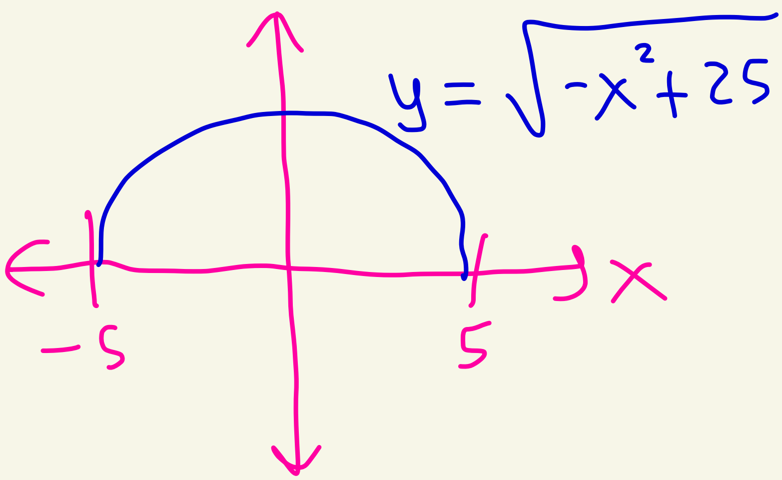
$$y^2 = -x^2 + 25$$



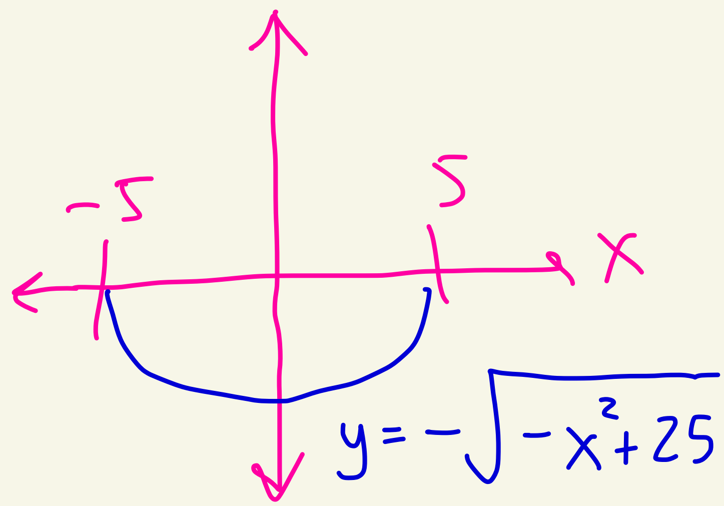
We have

$$y = \pm\sqrt{-x^2+25}$$

Do we pick + or − ?



$$y(4) = \sqrt{-(4)^2+25}$$
$$= \sqrt{9} = 3$$



$$y(4) = -\sqrt{-(4)^2+25}$$
$$= -\sqrt{9} = -3$$

We want $y(4)=3$.

Answer

So we want $y=\sqrt{-x^2+25}$

This function is defined for

$-5 \leq x \leq 5$

Topic 5 – First order Exact Equations

Suppose you have a first order equation of the form:

$$\underbrace{M(x,y)}_{} + \underbrace{N(x,y)}_{} \cdot y' = 0$$

expressions with #'s, x's, y's
no y' in them

Ex: $\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)}y' = 0$

Suppose that there exists a function $f(x,y)$ where

$$\frac{\partial f}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x,y)$$

Ex: $\underbrace{2xy}_{M} + \underbrace{(x^2-1)}_{N} y' = 0$

$M(x,y) = 2xy \,, \quad N(x,y) = x^2 - 1$

Let $f(x,y) = x^2 y - y$

Then,

$$\frac{\partial f}{\partial x} = 2xy - 0 = 2xy = M(x,y)$$

$$\frac{\partial f}{\partial y} = x^2 - 1 = N(x,y)$$

Continuing on...

Suppose $\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$

Then,

$$M(x,y) + N(x,y) \cdot y' = 0$$

becomes

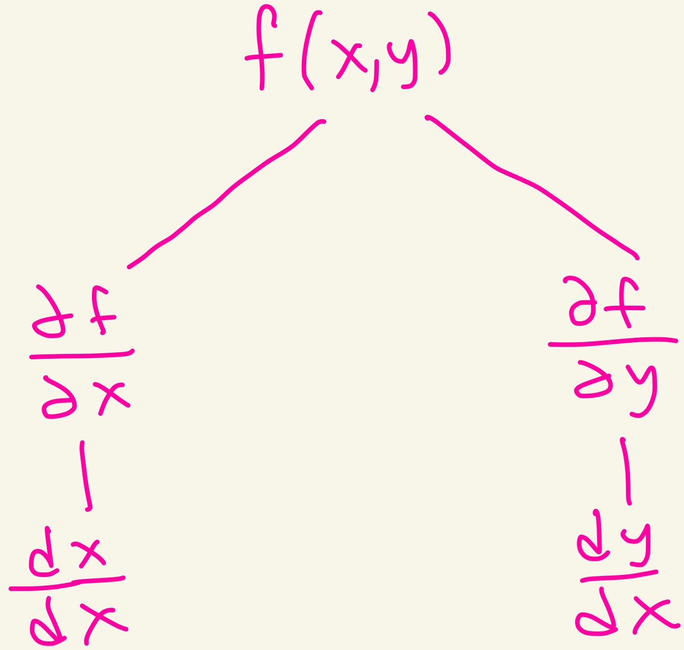
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$$

Recall from Math 2130

$f(x,y)$ is a function of x and y

$y = y(x)$ is a function of x

The chain rule says:



$$\frac{df}{dx} = \frac{\partial f}{\partial x} \cdot \frac{d}{dx}(x) + \frac{\partial f}{\partial y} \cdot \frac{d}{dx}(y)$$

$$= \frac{\partial f}{\partial x} \cdot 1 + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

Above we had

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$$

From the Calc III analysis this becomes

$$\frac{df}{dx} = 0$$

we want this to be satisfied

So, for example the family of curves given by $f(x,y) = c$ where c is a constant will satisfy

$$\frac{df}{dx} = 0.$$

<u>Summary:</u> If $\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$, then the equation $f(x,y) = c$ where c is any constant will give an implicit solution to

$$M(x,y) + N(x,y) \cdot y' = 0$$

If such an f exists then we say that

$$M(x,y) + N(x,y) \cdot y' = 0$$

is an <u>exact</u> equation.

Ex: Consider

$$\underbrace{2xy}_{M} + \underbrace{(x^2-1)}_{N} y' = 0$$

Let $f(x,y) = x^2 y - y$.

We saw

$$\frac{\partial f}{\partial x} = 2xy = M(x,y)$$

$$\frac{\partial f}{\partial y} = x^2 - 1 = N(x,y)$$

So, $2xy + (x^2-1)y' = 0$ is exact and a solution is given by $\underbrace{x^2 y - y = C}_{f(x,y)=C}$ where C is any constant.

Let's verify that $x^2y - y = c$ solves $2xy + (x^2-1)y' = 0$.

Check #1:

Differentiate

$$x^2y - y = c$$

with respect to x to get:

$$2xy + x^2\underbrace{y'}_{} - \underbrace{y'}_{} = 0$$

$$\frac{dy}{dx}$$

This is

$$2xy + (x^2-1)y' = 0$$

which is the equation we wanted to solve.

check #2:

Solve $x^2y - y = c$ with respect to y to get:

$$(x^2-1)y = c$$

$$y = \frac{c}{(x^2-1)}$$

$$y = c(x^2-1)^{-1}$$

Let's plug this into $2xy + (x^2-1)y' = 0$ to see if it works.

We have

$$y = c(x^2-1)^{-1} = \frac{c}{x^2-1}$$

$$y' = -c(x^2-1)^{-2} \cdot (2x) = \frac{-2xc}{(x^2-1)^2}$$

Plugging into left-side we get

$$2xy + (x^2-1)y'$$

$$= 2x\left(\frac{c}{x^2-1}\right) + (x^2-1)\left(\frac{-2xc}{(x^2-1)^2}\right)$$

$$= \frac{2xc}{x^2-1} - \frac{2xc}{x^2-1} = 0$$

It works!