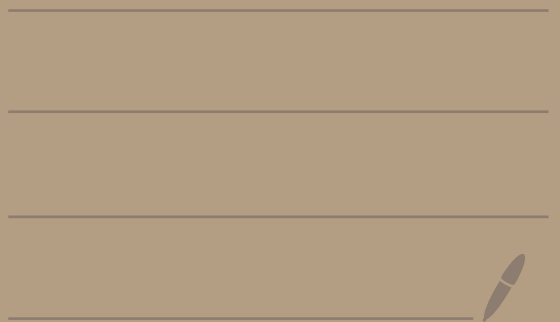


Math 2150-01

9/3/25



(Topic 3 continued...)

Ex: (HW 3-2(b))

Find all solutions to

$$x^2 y' + x(x+2)y = e^x$$

on $I = (0, \infty)$

want solution
to exist
when $0 < x$

Step 1: Divide by x^2 to
put the equation in the
proper form. We get

$$y' + \frac{x(x+2)}{x^2} y = \frac{e^x}{x^2}$$

This becomes

$$y' + \underbrace{\left(1 + \frac{z}{x}\right)}_{A(x)} y = \frac{e^x}{x^2}$$

Step 2: Do what we did last week.

Let

$$A(x) = \int \left(1 + \frac{z}{x}\right) dx$$

$$= x + 2 \ln|x|$$

$$= x + 2 \ln(x)$$

$x > 0$

since

$I = (0, \infty)$

We will next multiply by

$$e^{A(x)} = e^{x + 2 \ln(x)}$$

$$= e^x e^{2 \ln(x)}$$

$A \ln(B)$

$= \ln(B^A)$

$$\begin{aligned}
 &= e^x e^{\ln(x^2)} \\
 &= e^x \cdot x^2 \\
 &= x^2 e^x
 \end{aligned}$$

$e^{\ln(A)} = A$

Now multiply $y' + (1 + \frac{2}{x})y = \frac{e^x}{x^2}$

by $e^{A(x)} = x^2 e^x$ to get:

$$x^2 e^x y' + x^2 e^x (1 + \frac{2}{x} y) = x^2 e^x \cdot \frac{e^x}{x^2}$$

always: $(e^{A(x)} \cdot y)'$

We get

$$(x^2 e^x \cdot y)' = e^x e^x$$

So,

$$(x^2 e^x \cdot y)' = e^{x+x}$$

$$(x^2 e^x \cdot y)' = e^{2x}$$

Integrate to get

$$x^2 e^x y = \int e^{2x} dx$$

$$\int e^{2x} dx = \frac{1}{2} \int e^u du$$

$$\begin{aligned} u &= 2x \\ du &= 2dx \\ \frac{1}{2} du &= dx \end{aligned}$$

$$= \frac{1}{2} e^u + C$$

$$= \frac{1}{2} e^{2x} + C$$

We get

$$x^2 e^x y = \frac{1}{2} e^{2x} + C$$

So,

$$y = \frac{1}{x^2 e^x} \cdot \frac{1}{2} e^{2x} + C \cdot \frac{1}{x^2 e^x}$$

Thus,

$$y = \frac{e^x}{2x^2} + \frac{C}{x^2 e^x}$$

Topic 4 - Separable first order ODEs

Def: A first order ODE is called separable if it is of the form

$$\underbrace{N(y)}_{\text{just } y's \text{ and } \#s} \cdot y' = \underbrace{M(x)}_{\text{just } x's \text{ and } \#s}$$

Ex: $\underbrace{y^2}_{N(y)} y' = \underbrace{x^3 + 3}_{M(x)} \leftarrow \text{Separable}$

Ex: Is $y' = \frac{x^2}{y}$ separable?

Yes, because you can multiply by y to get

$$\underbrace{y}_{N(y)} \cdot y' = \underbrace{x^2}_{M(x)}$$

How to solve a separable ODE

Formal notation

$$N(y) \cdot y' = M(x)$$



$$N(y(x)) \cdot y'(x) = M(x)$$



$$\int N(y(x)) \cdot y'(x) dx = \int M(x) dx$$



$$\begin{aligned} u &= y(x) \\ du &= y'(x) dx \end{aligned}$$

$$\int N(u) du = \int M(x) dx$$

now integrate
Here $u = y$

Informal notation

$$N(y) \cdot \frac{dy}{dx} = M(x)$$



$$N(y) dy = M(x) dx$$

differential form
notation



$$\int N(y) dy = \int M(x) dx$$

Now integrate

Ex: Solve the initial-value problem

$$y^2 \frac{dy}{dx} = x - 5$$

$$y(0) = 1$$

Furthermore, give an interval where the solution exists.

We have

$$y^2 \frac{dy}{dx} = x - 5$$

$$y^2 dy = (x - 5) dx$$

$$\int y^2 dy = \int (x - 5) dx$$

← separable
←

$$\frac{1}{3}y^3 = \frac{1}{2}x^2 - 5x + C$$

Before we solve for y let's plug in the $y(0)=1$ condition. This means when $x=0$ we need $y=1$. So we get:

$$\frac{1}{3}(1)^3 = \frac{1}{2}(0)^2 - 5(0) + C$$

$$\frac{1}{3} = C$$

So,

$$\frac{1}{3}y^3 = \frac{1}{2}x^2 - 5x + \frac{1}{3}$$

Thus,

$$y = \left(\frac{3}{2}x^2 - 15x + 1 \right)^{1/3}$$

This solution is defined
on $I = (-\infty, \infty)$

