Math 2150-01
9/3/25

(Topic 3 continued...)

Ex: (HW 3-2(b))

Find all solutions to

$$x^2 y' + x(x+2)y = e^x$$

on $I = (0, \infty)$ ← want solution to exist when $0 < x$

Step 1: Divide by x^2 to put the equation in the proper form. We get

$$y' + \frac{x(x+2)}{x^2} y = \frac{e^x}{x^2}$$

This becomes

$$y' + \underbrace{\left(1+\frac{2}{x}\right)}y = \frac{e^x}{x^2}$$

Step 2: Do what we did last week.

Let

$$A(x) = \int \left(1+\frac{2}{x}\right)dx$$

$$= x + 2\ln|x|$$

$x>0$ since $I=(0,\infty)$

$$= x + 2\ln(x)$$

We will next multiply by

$$e^{A(x)} = e^{x+2\ln(x)}$$

$$= e^x e^{2\ln(x)}$$

$A\ln(B)$
$= \ln(B^A)$

$$= e^x e^{\ln(x^2)}$$

$$= e^x \cdot x^2$$

$$e^{\ln(A)} = A$$

$$= x^2 e^x$$

Now multiply $y' + \left(1+\frac{2}{x}\right)y = \frac{e^x}{x^2}$

by $e^{A(x)} = x^2 e^x$ to get:

$$\underbrace{x^2 e^x y' + x^2 e^x\left(1+\frac{2}{x}y\right)}_{\text{always: } \left(e^{A(x)}\cdot y\right)'} = x^2 e^x \cdot \frac{e^x}{x^2}$$

We get

$$\left(x^2 e^x \cdot y\right)' = e^x e^x$$

So,

$$(x^2 e^x \cdot y)' = e^{x+x}$$

$$(x^2 e^x \cdot y)' = e^{2x}$$

Integrate to get

$$x^2 e^x y = \underbrace{\int e^{2x}\,dx}$$

$$\int e^{2x}\,dx = \frac{1}{2}\int e^u\,du$$

$$\boxed{\begin{aligned} u &= 2x \\ du &= 2\,dx \\ \tfrac{1}{2}\,du &= dx \end{aligned}}$$

$$= \frac{1}{2}e^u + C$$

$$= \frac{1}{2}e^{2x} + C$$

We get

$$x^2 e^x y = \frac{1}{2} e^{2x} + C$$

So,

$$y = \frac{1}{x^2 e^x} \cdot \frac{1}{2} e^{2x} + C \cdot \frac{1}{x^2 e^x}$$

Thus,

$$y = \frac{e^x}{2x^2} + \frac{C}{x^2 e^x}$$

Topic 4 - Separable first order ODEs

Def: A first order ODE is called <u>separable</u> if it is of the form

$$\underbrace{N(y)}_{\text{just } y\text{'s and \#s}} \cdot y' = \underbrace{M(x)}_{\text{just } x\text{'s and \#s}}$$

Ex: $\underbrace{y^2}_{N(y)} y' = \underbrace{x^3+3}_{M(x)} \leftarrow$ Separable

Ex: Is $y' = \frac{x^2}{y}$ separable?

Yes, because you can multiply by y to get

$$\underbrace{y}_{N(y)} \cdot y' = \underbrace{x^2}_{M(x)}$$

How to solve a separable ODE

Formal notation

$$N(y)\cdot y' = M(x)$$

$$\downarrow$$

$$N(y(x))\cdot y'(x) = M(x)$$

$$\downarrow$$

$$\int N(y(x))\cdot y'(x)\,dx = \int M(x)\,dx$$

$$\downarrow$$

$u = y(x)$
$du = y'(x)\,dx$

$$\int N(u)\,du = \int M(x)\,dx$$

now integrate

Here $u = y$

Informal notation

$$N(y)\cdot \frac{dy}{dx} = M(x)$$

$$\downarrow$$

$$N(y)\,dy = M(x)\,dx$$

differential form notation

$$\downarrow$$

$$\int N(y)\,dy = \int M(x)\,dx$$

Now integrate

Ex: Solve the initial-value problem

$$\boxed{\begin{gathered} y^2 \frac{dy}{dx} = x - 5 \\ y(0) = 1 \end{gathered}}$$

Furthermore, give an interval where the solution exists.

We have

$$y^2 \frac{dy}{dx} = x - 5$$

separable

$$y^2 \, dy = (x-5) \, dx$$

$$\int y^2 \, dy = \int (x-5) \, dx$$

$$\frac{1}{3}y^3 = \frac{1}{2}x^2 - 5x + C$$

Before we solve for y let's plug in the $y(0)=1$ condition. This means when $x=0$ we need $y=1$. So we get:

$$\frac{1}{3}(1)^3 = \frac{1}{2}(0)^2 - 5(0) + C$$

$$\frac{1}{3} = C$$

So,

$$\frac{1}{3}y^3 = \frac{1}{2}x^2 - 5x + \frac{1}{3}$$

Thus,

$$y = \left(\frac{3}{2}x^2 - 15x + 1\right)^{1/3}$$

This solution is defined on $I = (-\infty, \infty)$