Math 2150-01
9/24/25

(Topic 7 continued...)

$\underline{\text{Ex:}}$ Solve

$$\boxed{\begin{aligned} &4y'' - y' = 0 \\ &y(0) = -1 \\ &y'(0) = 1 \end{aligned}}$$

The characteristic polynomial is

$$4r^2 - r = 0$$

$$r(4r-1) = 0$$

Either $r = 0$ or $4r - 1 = 0$

roots: $r = 0, \frac{1}{4}$

The general solution of

$$4y'' - y' = 0 \text{ is}$$

$$y_h = c_1 e^{0x} + c_2 e^{\frac{1}{4}x}$$

$$\boxed{y_h = c_1 + c_2 e^{x/4}}$$

We want

$$y_h(0) = -1$$

$$y_h'(0) = 1$$

We have

$$y_h = c_1 + c_2 e^{x/4}$$

$$y_h' = \frac{1}{4} c_2 e^{x/4}$$

We want

$$\boxed{\begin{aligned} -1 &= c_1 + c_2 e^{0/4} \\ 1 &= \tfrac{1}{4} c_2 e^{0/4} \end{aligned}}$$

$y_h(0) = -1$

$y_h'(0) = 1$

$$\boxed{\begin{aligned} -1 &= c_1 + c_2 \quad (1) \\ 1 &= \tfrac{1}{4} c_2 \quad (2) \end{aligned}}$$

(2) gives $c_2 = 4$.

Plug into (1) to get $c_1 = -1 - c_2$

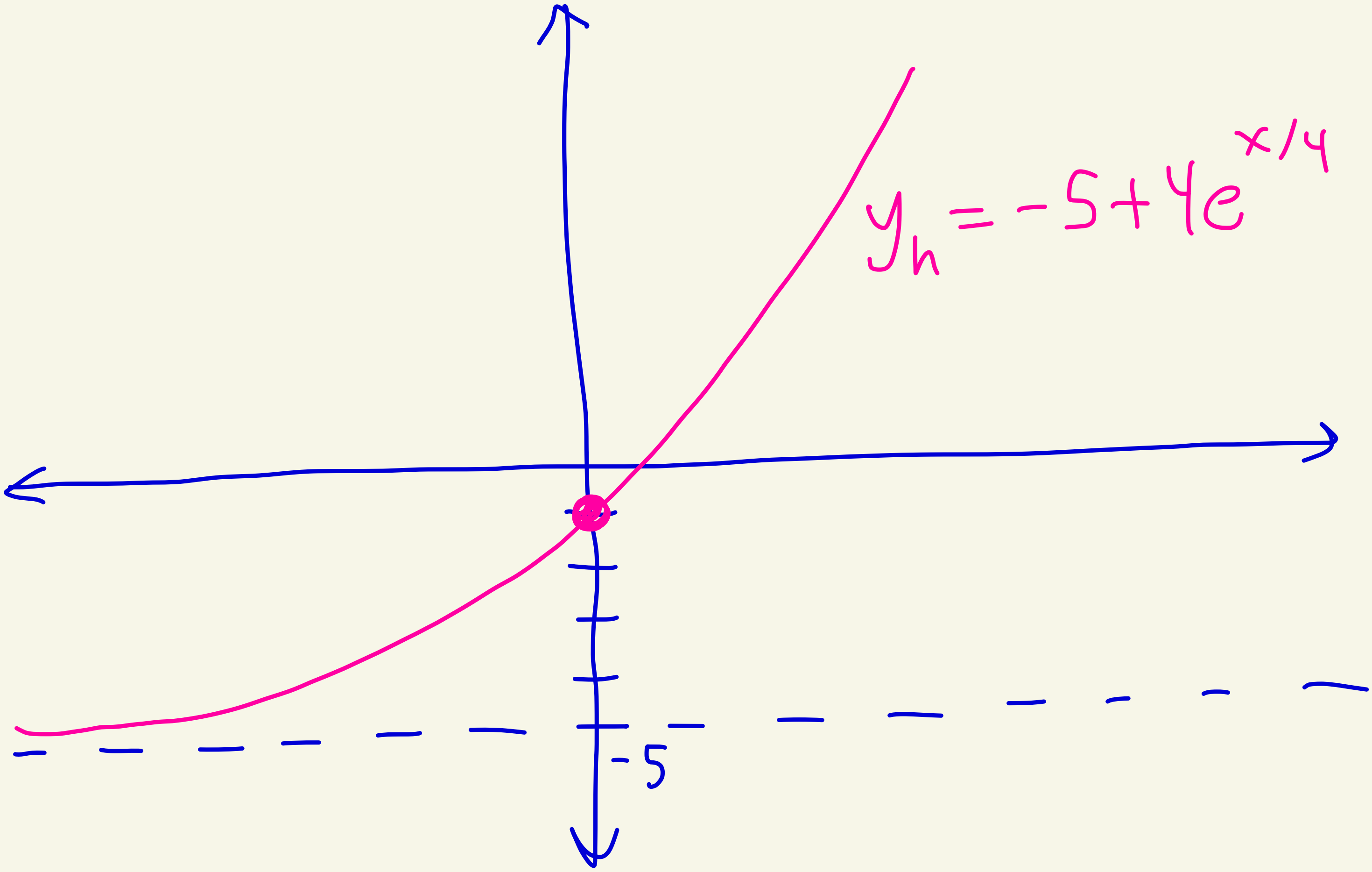
$= -1 - 4 = -5$.

The solution to the IVP

$4y'' - y' = 0, \quad y(0) = -1, \; y'(0) = 1$

is

$y_h = -5 + 4e^{x/4}$



Why do the formulas we gave last time work?

Suppose we are looking at

$$\boxed{a_2 y'' + a_1 y' + a_0 y = 0 \quad (*)}$$

with characteristic polynomial

$$a_2 r^2 + a_1 r + a_0 = 0$$

Suppose there is a real root $\hat{r}$ (so we are in case 1 or case 2).

Let $y = e^{\hat{r}x}$.

Let's show this is a solution to $(*)$.

We have

$$y = e^{\hat{r}x}, \quad y' = \hat{r}e^{\hat{r}x}, \quad y'' = \hat{r}^2 e^{\hat{r}x}$$

Then, plugging into the LHS of (*) gives

$$a_2 y'' + a_1 y' + a_0 y$$

$$= a_2 \hat{r}^2 e^{\hat{r}x} + a_1 \hat{r} e^{\hat{r}x} + a_0 e^{\hat{r}x}$$

$$= e^{\hat{r}x} \underbrace{\left(a_2 \hat{r}^2 + a_1 \hat{r} + a_0\right)}_{\text{characteristic poly with } \hat{r} \text{ plugged in}}$$

$$= e^{\hat{r}x}(0) = 0$$

Thus, if $\hat{r}$ is a root of the characteristic polynomial then $y = e^{\hat{r}x}$ will solve $a_2 y'' + a_1 y' + a_0 = 0$.

Suppose now we are in case 1 where the characteristic polynomial has two distinct real roots r_1, r_2 ($r_1 \neq r_2$).

Then by the argument above we get two solutions to (*) given by $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$.

Ex: $y'' - 3y' + 2y = 0$

$$r^2 - 3r + 2 = 0$$

$$(r-2)(r-1) = 0$$

$$r_1 = 2, \ r_2 = 1$$

$$y_1 = e^{2x}, \quad y_2 = e^{x}$$

Let's show $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are linearly independent

We have

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix}$$

$$= (e^{r_1 x})(r_2 e^{r_2 x}) - (e^{r_2 x})(r_1 e^{r_1 x})$$

$$= r_2 e^{(r_1 + r_2)x} - r_1 e^{(r_1 + r_2)x}$$

$$= \underbrace{(r_2 - r_1)}_{\substack{r_2 - r_1 \neq 0 \\ \text{since } r_1 \neq r_2}} \underbrace{e^{(r_1 + r_2)x}}_{\text{never } 0}$$

So, $W(y_1, y_2)$ is never 0.

It's not the zero function.

Thus, $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$ for all x.

And for case 1, the solution to $(*)$ is

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$