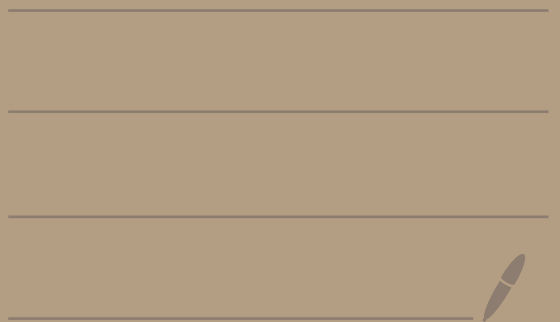


Math 2150-01

9/22/25



Last time we saw that
the general solution to

$$y'' - 7y' + 10y = 24e^x$$

on $I = (-\infty, \infty)$ is

$$y = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{\text{general solution } y_h \text{ to } y'' - 7y' + 10y = 0} + \underbrace{6e^x}_{\text{particular solution } y_p \text{ to } y'' - 7y' + 10y = 24e^x}$$

So we get an infinite #
of solutions to

$$y'' - 7y' + 10y = 24e^x$$

one for each choice of c_1 & c_2 .

If we specify $y(x_0)$ and $y'(x_0)$ at some x_0 then we will have an initial-value problem with just one solution.

Ex: Solve

$$y'' - 7y' + 10y = 24e^x$$

$$y(0) = 0, \quad y'(0) = 1$$


$$x_0 = 0$$

We know a solution to

$$y'' - 7y' + 10y = 24e^x$$

must be of the form

$$y = c_1 e^{2x} + c_2 e^{5x} + 6e^x$$

Let's make it solve

$$y(0) = 0 \text{ and } y'(0) = 1$$

We have

$$y = c_1 e^{2x} + c_2 e^{5x} + 6e^x$$

$$y' = 2c_1 e^{2x} + 5c_2 e^{5x} + 6e^x$$

We have

$$\begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases}$$

$$\begin{cases} c_1 e^{2(0)} + c_2 e^{5(0)} + 6e^0 = 0 \\ 2c_1 e^{2(0)} + 5c_2 e^{5(0)} + 6e^0 = 1 \end{cases}$$

1

$$\downarrow \quad e^0 = 1$$

$$\begin{array}{l} c_1 + c_2 + 6 = 0 \quad (1) \\ 2c_1 + 5c_2 + 6 = 1 \quad (2) \end{array}$$



$$\begin{array}{l} c_1 + c_2 = -6 \quad (1) \\ 2c_1 + 5c_2 = -5 \quad (2) \end{array}$$

① gives $c_1 = -6 - c_2$

Plug into ② to get

$$2(-6 - c_2) + 5c_2 = -5$$

So,

$$3c_2 = 7$$

Thus, $c_2 = 7/3$.

$$\begin{aligned}\text{So, } c_1 &= -6 - c_2 = -6 - \frac{7}{3} \\ &= -\frac{18}{3} - \frac{7}{3} \\ &= -\frac{25}{3}\end{aligned}$$

The answer is:

$$y = \frac{-25}{3} e^{2x} + \frac{7}{3} e^{5x} + 6e^x$$

c_1

c_2

This is the unique solution to
 $y'' - 7y' + 10y = 24e^x$
 $y(0) = 0, y'(0) = 1$

Topic 7 - 2nd order linear homogeneous constant coefficient ODEs

We now learn how to solve

$$a_2 y'' + a_1 y' + a_0 y = 0$$

Where a_2, a_1, a_0 are constants.

Ex: $y'' - 7y' + 10y = 0$

$a_2 = 1$

$a_1 = -7$

$a_0 = 10$

Def: The characteristic equation of

$$a_2 y'' + a_1 y' + a_0 y = 0$$

is

$$a_2 r^2 + a_1 r + a_0 = 0$$

where r is a number/variable

Ex: The characteristic equation of

$$y'' - 7y' + 10y = 0$$

is

$$r^2 - 7r + 10 = 0$$

Formula time

Consider

$$a_2 y'' + a_1 y' + a_0 y = 0 \quad (*)$$

where a_2, a_1, a_0 are constants
and assume $a_2 \neq 0$.

There are three cases, depending
on the roots of the
characteristic equation

$$a_2 r^2 + a_1 r + a_0 = 0.$$

Case 1: If the characteristic
equation has two distinct
real roots r_1, r_2 ($r_1 \neq r_2$)
then the solution to (*) is

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Case 2: If the characteristic equation has a repeated real root r , then the solution to (*) is

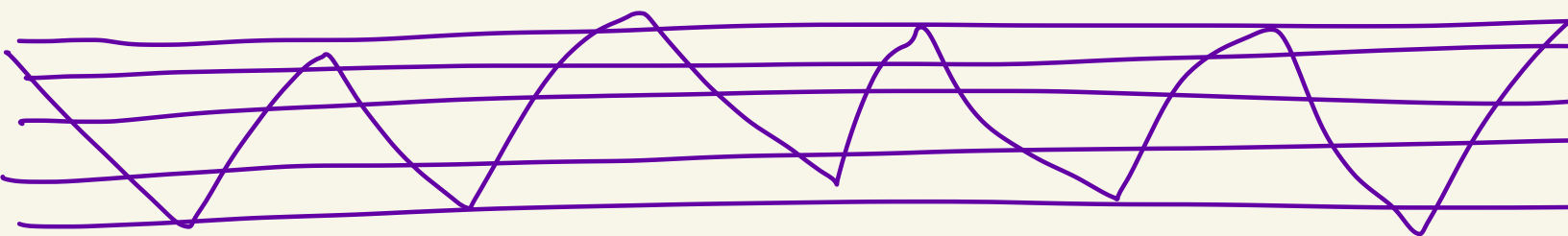
$$y_h = c_1 e^{rx} + c_2 x e^{rx}$$

Case 3: If the characteristic equation has imaginary/complex roots $\alpha \pm i\beta$ ←

then the solution to (*) is

$$\begin{aligned}\alpha &= \text{alpha} \\ \beta &= \text{beta} \\ i &= \sqrt{-1}\end{aligned}$$

$$y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$



Ex: Solve $y'' - 7y' + 10y = 0$.

The characteristic equation is

$$r^2 - 7r + 10 = 0$$

The roots are

$$r = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(10)}}{2(1)}$$

$$= \frac{7 \pm \sqrt{9}}{2} = \frac{7 \pm 3}{2}$$

$$= \frac{7+3}{2}, \frac{7-3}{2} = \boxed{5, 2}$$

↑
two real roots
case 1

Thus the solution is

$$y_h = c_1 e^{2x} + c_2 e^{5x}$$

Ex: Solve $y'' - 4y' + 4y = 0$

The characteristic equation is

$$r^2 - 4r + 4 = 0$$

The roots are

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{0}}{2} = \frac{4}{2} = \boxed{2}$$

The solution is

$$y_h = c_1 e^{2x} + c_2 x e^{2x}$$

repeated
real
root
case 2

Ex: Solve $y'' - 4y' + 13y = 0$

The characteristic equation is

$$r^2 - 4r + 13 = 0$$

The roots are

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm \sqrt{36} \sqrt{-1}}{2}$$

$$= \frac{4 \pm 6i}{2} = \frac{4}{2} \pm \frac{6}{2}i$$

↑
 $\boxed{\sqrt{-1} = i}$

$$= \frac{2 \pm 3i}{1}$$

$$\alpha \pm \beta i$$

$$\alpha = 2, \beta = 3$$

Answer is

$$y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$
$$= c_1 e^{2x} \cos(3x) + c_2 e^{2x} \sin(3x)$$
