Math 2150-01
9/22/25

Last time we saw that the general solution to

$$y'' - 7y' + 10y = 24e^{x}$$

on $I = (-\infty, \infty)$ is

$$y = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{\substack{\text{general solution} \\ y_h \text{ to} \\ y''-7y'+10y=0}} + \underbrace{6e^{x}}_{\substack{\text{particular solution} \\ y_p \\ \text{to} \\ y''-7y'+10y=24e^x}}$$

So we get an infinite # of solutions to

$$y'' - 7y' + 10y = 24e^{x}$$

one for each choice of c_1 & c_2.

If we specify $y(x_0)$ and $y'(x_0)$ at some x_0 then we will have an initial-value problem with just one solution.

Ex: Solve

$$y'' - 7y' + 10y = 24e^x$$

$$y(0) = 0, \quad y'(0) = 1$$

$x_0 = 0$

We know a solution to

$$y'' - 7y' + 10y = 24e^x$$

must be of the form

$$y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$$

Let's make it solve

$y(0) = 0$ and $y'(0) = 1$

We have

$$y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$$

$$y' = 2c_1 e^{2x} + 5c_2 e^{5x} + 6e^{x}$$

We have

$$\boxed{\begin{aligned} y(0) &= 0 \\ y'(0) &= 1 \end{aligned}}$$

$$\boxed{\begin{aligned} c_1 e^{2(0)} + c_2 e^{5(0)} + 6e^{0} &= 0 \\ 2c_1 e^{2(0)} + 5c_2 e^{5(0)} + 6e^{0} &= 1 \end{aligned}}$$

$e^0 = 1$

$$c_1 + c_2 + 6 = 0 \quad (1)$$
$$2c_1 + 5c_2 + 6 = 1 \quad (2)$$

$$c_1 + c_2 = -6 \quad (1)$$
$$2c_1 + 5c_2 = -5 \quad (2)$$

(1) gives $c_1 = -6 - c_2$

Plug into (2) to get

$$2(-6 - c_2) + 5c_2 = -5$$

So,

$$3c_2 = 7$$

Thus, $c_2 = 7/3$.

So, $c_1 = -6 - c_2 = -6 - \frac{7}{3}$

$$= \frac{-18}{3} - \frac{7}{3}$$

$$= \frac{-25}{3}$$

$$\boxed{\text{The answer is: } y = \underbrace{\frac{-25}{3}}_{c_1} e^{2x} + \underbrace{\frac{7}{3}}_{c_2} e^{5x} + 6e^{x}}$$

This is the unique solution to

$$y'' - 7y' + 10y = 24e^x$$

$$y(0) = 0, \quad y'(0) = 1$$

$$\boxed{\text{Topic 7 – 2nd order linear homogeneous constant coefficient ODEs}}$$

We now learn how to solve

$$a_2 y'' + a_1 y' + a_0 y = 0$$

where a_2, a_1, a_0 are constants.

Ex: $y'' - 7y' + 10y = 0$

$\boxed{a_2 = 1}$ $\boxed{a_1 = -7}$ $\boxed{a_0 = 10}$

Def: The characteristic equation of

$$a_2 y'' + a_1 y' + a_0 y = 0$$

is

$$a_2 r^2 + a_1 r + a_0 = 0$$

where r is a number/variable

Ex: The characteristic equation of

$$y'' - 7y' + 10y = 0$$

is

$$r^2 - 7r + 10 = 0$$

Formula time

Consider

$$\boxed{a_2 y'' + a_1 y' + a_0 y = 0 \qquad (*)}$$

where a_2, a_1, a_0 are constants

and assume $a_2 \neq 0$.

There are three cases, depending on the roots of the characteristic equation

$$a_2 r^2 + a_1 r + a_0 = 0.$$

<u>Case 1</u>: If the characteristic equation has two distinct real roots r_1, r_2 $(r_1 \neq r_2)$ then the solution to $(*)$ is

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

<u>Case 2:</u> If the characteristic equation has a repeated real root r, then the solution to $(*)$ is

$$y_h = c_1 e^{rx} + c_2 x e^{rx}$$

<u>Case 3:</u> If the characteristic equation has imaginary/complex roots $\alpha \pm i\beta$ then the solution to $(*)$ is

α = alpha
β = beta
$i = \sqrt{-1}$

$$\boxed{y_h = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)}$$

Ex: Solve $y'' - 7y' + 10y = 0$.

The characteristic equation is

$$r^2 - 7r + 10 = 0$$

The roots are

$$r = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(10)}}{2(1)}$$

$$= \frac{7 \pm \sqrt{9}}{2} = \frac{7 \pm 3}{2}$$

$$= \frac{7+3}{2}, \frac{7-3}{2} = \boxed{5, 2}$$

two real roots

case 1

Thus the solution is

$$y_h = c_1 e^{2x} + c_2 e^{5x}$$

<u>Ex:</u> Solve $y'' - 4y' + 4y = 0$

The characteristic equation is

$$r^2 - 4r + 4 = 0$$

The roots are

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{0}}{2} = \frac{4}{2} = \boxed{2}$$

The solution is

$$y_h = c_1 e^{2x} + c_2 x e^{2x}$$

repeated real root case 2

Ex: Solve $y'' - 4y' + 13y = 0$

The characteristic equation is

$$r^2 - 4r + 13 = 0$$

The roots are

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm \sqrt{36}\sqrt{-1}}{2}$$

$$= \frac{4 \pm 6i}{2} = \frac{4}{2} \pm \frac{6}{2} i$$

$\boxed{\sqrt{-1} = i}$

$$= \underbrace{2 \pm 3i}_{\alpha \pm \beta i}$$

$\alpha = 2, \beta = 3$

Answer is

$$y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

$$= c_1 e^{2x} \cos(3x) + c_2 e^{2x} \sin(3x)$$