Math 2150-01

9/17/25

Ex: Let's solve

$$y'' - 7y' + 10y = 24e^{x}$$

on $I = (-\infty, \infty)$

Step 1: Find the general solution y_h to

$$y'' - 7y' + 10y = 0$$

homogeneous equation

We need two linearly independent solutions.

Let $f_1(x) = e^{2x}$, $f_2(x) = e^{5x}$.

Last time we showed that f_1 and f_2 are linearly independent. Let's show they both solve

$$y'' - 7y' + 10y = 0.$$

Let's check f_1 first.

$f_1(x) = e^{2x}$, $f_1'(x) = 2e^{2x}$, $f_1''(x) = 4e^{2x}$

Plug it in to get

$$\begin{aligned} f_1'' - 7f_1' + 10f_1 &= 4e^{2x} - 7\left(2e^{2x}\right) + 10\left(e^{2x}\right) \\ &= 4e^{2x} - 14e^{2x} + 10e^{2x} \\ &= 0 \end{aligned}$$

It works!

Let's check $f_2(x) = e^{5x}$.

$f_2(x) = e^{5x}$, $f_2'(x) = 5e^{5x}$, $f_2''(x) = 25e^{5x}$

Thus,

$$\begin{aligned} f_2'' - 7f_1' + 10f_1 &= 25e^{5x} - 7(5e^{5x}) + 10(e^{5x}) \\ &= 25e^{5x} - 35e^{5x} + 10e^{5x} \\ &= 0 \end{aligned}$$

So, f_2 is also a solution.

Since $f_1(x) = e^{2x}$ and $f_2(x) = e^{5x}$ are linearly independent solutions to

$$y'' - 7y' + 10y = 0$$

We know that every solution to $y'' - 7y' + 10y = 0$ is of the form

$$y_h = c_1 e^{2x} + c_2 e^{5x}$$

where c_1, c_2 can be any constants.

Some example solutions to $y'' - 7y' + 10y = 0$ are

$\underline{c_1 = 1,\ c_2 = 1:}$ $y_h = e^{2x} + e^{5x}$

$\underline{c_1 = \frac{1}{2},\ c_2 = \sqrt{2}:}$ $y_h = \frac{1}{2} e^{2x} + \sqrt{2}\, e^{5x}$

<u>Step 2:</u> Let's solve

$$y'' - 7y' + 10y = 24e^{x}$$

On $I=(-\infty,\infty)$

We need a particular solution y_p to $y''-7y'+10y=24e^x$.

Let's try

$$y_p = 6e^x$$

in topic 8 we see how to find this

Plug it in to see if it solves the ODE.

$y_p = 6e^x$, $y_p' = 6e^x$, $y_p'' = 6e^x$

We have

$$y_p'' - 7y_p' + 10y_p = 6e^x - 7(6e^x) + 10(6e^x)$$
$$= 6e^x - 42e^x + 60e^x$$

$= 24e^{x}$

So, $y_p = 6e^{x}$ solves $y'' - 7y' + 10y = 24e^{x}$.

Then every solution of

$$y'' - 7y' + 10y = 24e^{x}$$

is of the form:

$$y = y_h + y_p = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{y_h} + \underbrace{6e^{x}}_{y_p}$$

where c_1, c_2 are any constants

Ex: Let's find all the solutions to

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

on $I = (0, \infty)$ ← $\boxed{x > 0}$

Step 1: Find two linearly independent solutions to

$$x^2 y'' - 4xy' + 6y = 0$$

homogeneous equation

Let

$$f_1(x) = x^2$$
$$f_2(x) = x^3$$

topic 10 shows to find these

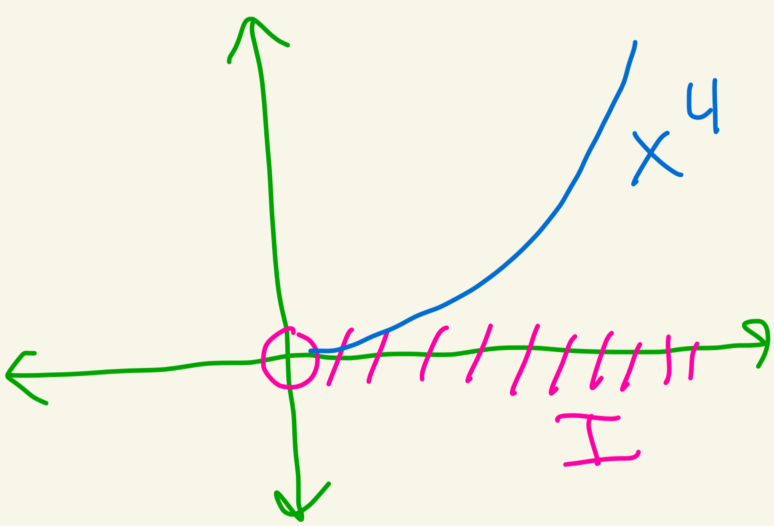
Let's first verify f_1 and f_2 are linearly independent on I.

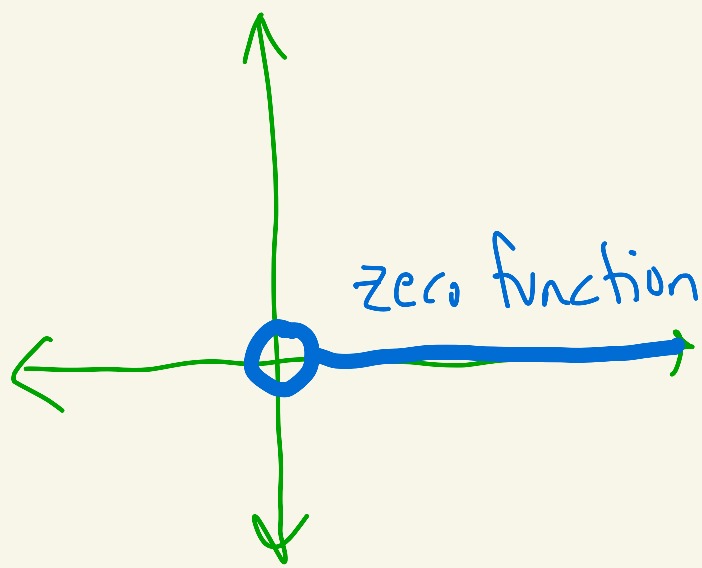
$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= (x^2)(3x^2) - (x^3)(2x)$$

$$= x^4$$

Is $W(f_1, f_2) = x^4$ the zero function on $I = (0, \infty)$?





They are not equal.

Thus, $f_1(x) = x^2$ and $f_2(x) = x^3$ are linearly independent on I.

Let's now show that f_1 and f_2 are both solutions to

$$x^2 y'' - 4xy' + 6y = 0$$

For f_1 we have: $f_1 = x^2$, $f_1' = 2x$, $f_1'' = 2$

So:

$$\begin{aligned} x^2 f_1'' - 4x f_1' + 6f_1 &= x^2(2) - 4x(2x) + 6(x^2) \\ &= 2x^2 - 8x^2 + 6x^2 \\ &= 0 \end{aligned}$$

So, f_1 is a solution.

For f_2 we get: $f_2 = x^3$, $f_2' = 3x^2$, $f_2'' = 6x$

So:

$$x^2 f_2'' - 4x f_2' + 6f_2$$
$$= x^2(6x) - 4x(3x^2) + 6(x^3)$$
$$= 6x^3 - 12x^3 + 6x^3$$
$$= 0$$

So, f_2 is a solution.

Since $f_1(x) = x^2$, $f_2(x) = x^3$ are linearly independent solutions to

$$x^2 y'' - 4xy' + 6y = 0$$

on $I = (0, \infty)$, every solution is of the form

$$y_h = c_1 x^2 + c_2 x^3$$

$\boxed{\text{where } c_1, c_2 \text{ are any constants}}$

Step 2: Now let's find a particular solution y_p to

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

on $I = (0, \infty)$.

Let $y_p = \dfrac{1}{12x} = \dfrac{1}{12} x^{-1}$ ← topic 9 shows how to find this

Let's plug it in.

$$y_p = \frac{1}{12} x^{-1}$$

$$y_p' = -\frac{1}{12} x^{-2}$$

$$y_p'' = -\frac{1}{12}\left(-2x^{-3}\right) = \frac{1}{6}x^{-3}$$

Plug it in to get:

$$x^2 y_p'' - 4xy_p' + 6y_p$$

$$= x^2\left(\frac{1}{6}x^{-3}\right) - 4x\left(-\frac{1}{12}x^{-2}\right) + 6\left(\frac{1}{12}x^{-1}\right)$$

$$= \frac{1}{6}x^{-1} + \frac{1}{3}x^{-1} + \frac{1}{2}x^{-1}$$

$$= \left(\frac{1+2+3}{6}\right)x^{-1} = x^{-1} = \frac{1}{x}$$

Thus, $y_p = \frac{1}{12}x^{-1}$ solves

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}.$$

The general solution to

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

is

$$y = \underbrace{c_1 x^2 + c_2 x^3}_{y_h} + \underbrace{\frac{1}{12} x^{-1}}_{y_p}$$

Where c_1, c_2 are any constants