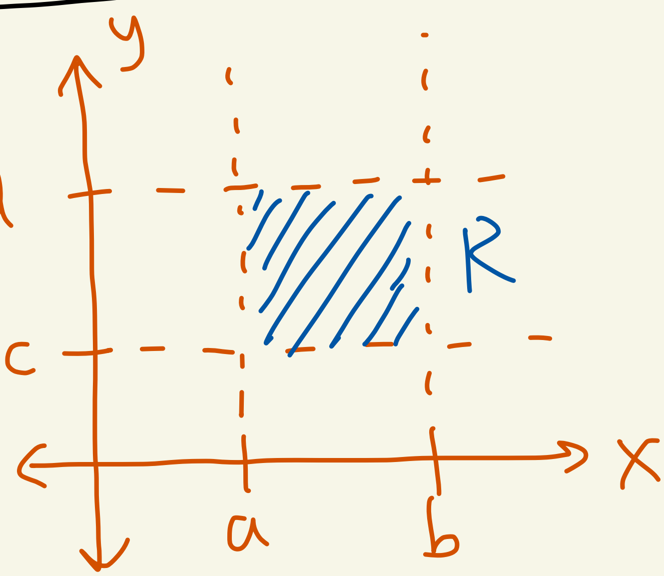
Math 2150-01
9/10/25

When does a function $f(x,y)$ exist with $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$?

Theorem: Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in some rectangle R defined by $a < x < b$, $c < y < d$.



Here a,b,c,d can be $\pm\infty$.

Then, there will exist an $f(x,y)$ with

$$\frac{\partial f}{\partial x} = M(x,y) \text{ and } \frac{\partial f}{\partial y} = N(x,y)$$

if and only if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}.$$

---

proof in online notes if want to see

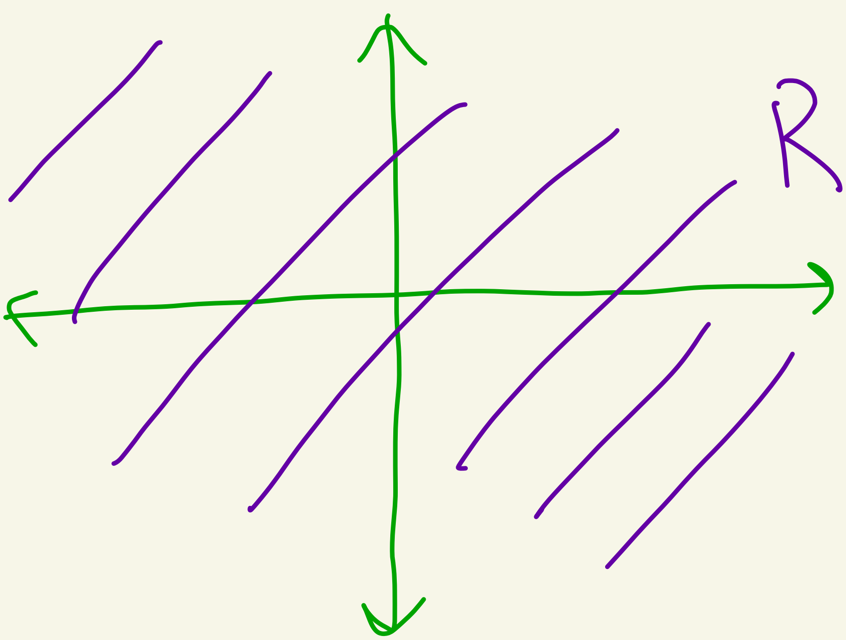
Ex: Consider

$$\underbrace{2xy}_{M} + \underbrace{(x^2-1)}_{N} y' = 0$$

So,

$$\left.\begin{aligned} M(x,y) &= 2xy \\ N(x,y) &= x^2-1 \end{aligned}\right\} \text{polynomials continuous everywhere}$$

$$\left.\begin{aligned} \frac{\partial M}{\partial x} &= 2y & \qquad \frac{\partial N}{\partial x} &= 2x \\ \frac{\partial M}{\partial y} &= 2x & \qquad \frac{\partial N}{\partial y} &= 0 \end{aligned}\right\} \text{polynomials continuous everywhere}$$



R can be the entire xy-plane for the theorem

Note that

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Thus, the ODE

$$2xy + (x^2-1)y' = 0$$

is exact, that is there exists $f(x,y)$ where

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 2xy \\ \frac{\partial f}{\partial y} &= x^2-1 \end{aligned}}$$

$\leftarrow \boxed{\frac{\partial f}{\partial x} = M}$

$\leftarrow \boxed{\frac{\partial f}{\partial y} = N}$

Last time I gave you the f
let's see how I found it.

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 2xy \quad (1) \\ \frac{\partial f}{\partial y} &= x^2 - 1 \quad (2) \end{aligned}}$$

Integrate (1) with respect to x to get:

$$\boxed{f(x,y) = x^2 y + \underbrace{C(y)}_{\text{constant w/ respect to x. Can have \#s and ys in it.}}}$$

Integrate (2) with respect to y to get:

$$\boxed{f(x,y) = x^2 y - y + \underbrace{D(x)}_{\text{constant w/ respect to y. Can have \#s and xs in it}}}$$

Set the above equal to each other to get:

$$\cancel{x^2 y} + C(y) = \cancel{x^2 y} - y + D(x)$$

So,

$$\underbrace{C(y)} = \underbrace{-y} + \underbrace{D(x)}_{0}$$

Set $C(y) = -y$ and $D(x) = 0$.

Plug either in above to find f.

For example,

$$f(x,y) = x^2 y + C(y)$$
$$= x^2 y - y$$

Ex: Find a solution to

$$\boxed{\begin{gathered}(e^x+y)+(2+x+ye^y)y'=0\\ y(0)=1\end{gathered}}$$

Let's see first: is this equation exact or not?

$$M(x,y)=e^x+y$$

$$N(x,y)=2+x+ye^y$$

$$\frac{\partial M}{\partial x}=e^x \qquad \frac{\partial N}{\partial x}=1$$

$$\frac{\partial M}{\partial y}=1 \qquad \frac{\partial N}{\partial y}=e^y+ye^y$$

continuous everywhere

We have: $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$

So,

$$(e^x + y) + (2 + x + ye^y)y' = 0$$

is exact.

Let's find $f(x,y)$ that solves:

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= e^x + y \quad (1) \\ \frac{\partial f}{\partial y} &= 2 + x + ye^y \quad (2) \end{aligned}}$$

$(1) \leftarrow \boxed{\frac{\partial f}{\partial x} = M}$

$(2) \leftarrow \boxed{\frac{\partial f}{\partial y} = N}$

Integrate (1) with respect to x:

$$\boxed{f(x,y) = e^x + yx + \underbrace{D(y)}_{\text{constant with respect to } x}}$$

Integrate (2) with respect to y:

$$\boxed{f(x,y) = 2y + xy + \underbrace{ye^y - e^y} + \underbrace{E(x)}_{\text{constant w/ respect to } x}}$$

$$\int ye^y\,dy = ye^y - \int e^y\,dy = ye^y - e^y$$

$$\boxed{\begin{array}{ll} u = y & du = dy \\ dv = e^y\,dy & v = e^y \\ \hline \int u\,dv = uv - \int v\,du & \end{array}}$$

Set the above equal to get:

$$e^x + \cancel{yx} + D(y) = 2y + \cancel{xy} + ye^y - e^y + E(x)$$

We get:

$$e^x + D(y) = 2y + ye^y - e^y + E(x)$$

Set

$$D(y) = 2y + ye^y - e^y$$

$$E(x) = e^x$$

Plug either one in to find f. We have

$$f(x,y) = e^x + yx + D(y)$$

$$f(x,y) = e^x + yx + 2y + ye^y - e^y$$

Thus a solution to

$$(e^x + y) + (2 + x + ye^y)\, y' = 0$$

is given by

$$\underbrace{e^x + yx + 2y + ye^y - e^y = c}_{f(x,y) = c}$$

where c is any constant

Let's now find a solution that satisfies $y(0) = 1$. Plug $x=0$, $y=1$ into our solution to get:

$$e^0 + (1)(0) + 2(1) + (1)e^1 - e^1 = c$$

$$3 = c$$

$$\boxed{\text{So an implicit solution is } e^x + yx + 2y + ye^y - e^y = 3}$$