Math 2150-01
8/27/25

(Topic 1 continued...)

Ex: Show that

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

is a solution to

$$y'' - 4y = 0$$

on $I = (-\infty, \infty)$

Here c_1, c_2 can be any constants.

In topic 7 we will learn how to find this solution

Ex: $c_1 = 4$, $c_2 = -\frac{1}{2}$

$$f(x) = 4e^{2x} - \frac{1}{2} e^{-2x}$$

Let's plug f into $y'' - 4y = 0$ and show that it solves it.

We get

$$\left.\begin{aligned} f(x) &= c_1 e^{2x} + c_2 e^{-2x} \\ f'(x) &= 2c_1 e^{2x} - 2c_2 e^{-2x} \\ f''(x) &= 4c_1 e^{2x} + 4c_2 e^{-2x} \end{aligned}\right] \begin{array}{l} \text{these} \\ \text{all} \\ \text{exist} \\ \text{on} \\ I = (-\infty, \infty) \end{array}$$

Plug in $y = f(x)$, $y'' = f''(x)$ to get

$$f''(x) - 4f(x)$$

$$= \left(4c_1 e^{2x} + 4c_2 e^{-2x}\right) - 4\left(c_1 e^{2x} + c_2 e^{-2x}\right)$$

$$= 0$$

So, f solves $y'' - 4y = 0$ on $I = (-\infty, \infty)$

Ex: Find c_1, c_2 where

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

solves the initial-value problem

$$\boxed{\begin{aligned} y'' - 4y &= 0 \\ y'(0) &= 0 \\ y(0) &= 1 \end{aligned}}$$

$\leftarrow$ ODE

initial conditions at $x_0 = 0$

We already know that $f(x) = c_1 e^{2x} + c_2 e^{-2x}$ solves $y'' - 4y = 0$.

Let's make it also satisfy $f(0) = 1$ and $f'(0) = 0$.

We have

$$\boxed{\begin{aligned} f(x) &= c_1 e^{2x} + c_2 e^{-2x} \\ f'(x) &= 2c_1 e^{2x} - 2c_2 e^{-2x} \end{aligned}}$$

$$\boxed{\begin{aligned} 1 &= c_1 e^{2(0)} + c_2 e^{-2(0)} \\ 0 &= 2c_1 e^{2(0)} - 2c_2 e^{-2(0)} \end{aligned}}$$

$1 = f(0)$

$0 = f'(0)$

$$\boxed{\begin{aligned} 1 &= c_1 + c_2 \\ 0 &= 2c_1 - 2c_2 \end{aligned}}$$

① ②

② gives $c_1 = c_2$.

Plug this into ① to get $1 = c_1 + c_1$.

So, $c_1 = 1/2$.

Then, $c_2 = c_1 = 1/2$.

Thus, $f(x) = \frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}$

is the solution we want.

Topic 3 – First order linear ODEs

We will give a method to solve

$$y' + a(x)y = b(x)$$

on any interval I where $a(x)$ and $b(x)$ are continuous.

Since $a(x)$ is continuous we can find an anti-derivative

$$A(x) = \int a(x)\,dx.$$

So, $A'(x) = a(x)$.

Now multiply $y' + a(x)y = b(x)$

by $e^{A(x)}$ to get:

$$\underbrace{e^{A(x)} y' + e^{A(x)} a(x)\, y}_{} = e^{A(x)} b(x)$$

this side becomes

$$\left(e^{A(x)} \cdot y\right)' = e^{A(x)} \cdot A'(x) \cdot y + e^{A(x)} y'$$
$$= e^{A(x)} \cdot a(x)\, y + e^{A(x)} y'$$

We have

$$\left(e^{A(x)} \cdot y\right)' = e^{A(x)} b(x)$$

Thus, by integrating both sides:

$$e^{A(x)} \cdot y = \int e^{A(x)} b(x)\, dx$$

Thus,

$$y = e^{-A(x)}\left[\int e^{A(x)} b(x)\,dx\right]$$

Since you can reverse the above steps this is the general solution to the ODE

Ex: Solve

$$y' + \underbrace{2x}_{a(x)=2x}\,y = \underbrace{x}_{b(x)=x}$$

on $I = (-\infty, \infty)$.

Let $A(x) = \int 2x\,dx = x^2$

no need for $+C$ here we just need one anti-derivative

multiply $y' + 2xy = x$

by $e^{A(x)} = e^{x^2}$ to get

$$\underbrace{e^{x^2} y' + e^{x^2}\cdot 2xy}_{\text{this is always } (e^{A(x)}\cdot y)'} = xe^{x^2}$$

We get

$$(e^{x^2}\cdot y)' = xe^{x^2}$$

So,

$$e^{x^2}\cdot y = \int xe^{x^2}\,dx$$

$$\int xe^{x^2}\,dx = \int \frac{1}{2}e^{u}\,du$$

$$\boxed{\begin{aligned} u &= x^2 \\ du &= 2x\,dx \\ \tfrac{1}{2}\,du &= x\,dx \end{aligned}}$$

$$= \frac{1}{2}e^{u} + C$$

$$= \frac{1}{2}e^{x^2} + C$$

Thus,

$$e^{x^2} \cdot y = \frac{1}{2}e^{x^2} + C$$

$$y = \underbrace{e^{-x^2}}_{\frac{1}{e^{x^2}}}\left(\frac{1}{2}e^{x^2} + C\right)$$

$$y = \frac{1}{2}e^{-x^2+x^2} + Ce^{-x^2}$$

$$y = \frac{1}{2} + Ce^{-x^2}$$

So the general solution to

$y' + 2xy = x$ is

$y = \frac{1}{2} + Ce^{-x^2}$ where C is any constant

Ex: Let's solve

$$y' + \cos(x)\, y = \sin(x)\cos(x)$$

on $I = (-\infty, \infty)$.

Let $A(x) = \int \cos(x)\,dx = \sin(x)$

Multiply the ODE above by $e^{A(x)} = e^{\sin(x)}$ to get:

$$\underbrace{e^{\sin(x)} y' + e^{\sin(x)} \cos(x)\, y}_{\text{this is always } \left(e^{A(x)} \cdot y\right)'} = e^{\sin(x)} \sin(x)\cos(x)$$

We get

$$\left(e^{\sin(x)} \cdot y\right)' = e^{\sin(x)} \cdot \sin(x)\cos(x)$$

So,

$$e^{\sin(x)} \cdot y = \int e^{\sin(x)} \cdot \sin(x)\cos(x)\,dx$$

$$\int e^{\sin(x)} \sin(x)\cos(x)\,dx = \int e^t \cdot t\,dt$$

$t = \sin(x)$
$dt = \cos(x)\,dx$

$$= \int t e^t\,dt = te^t - \int e^t\,dt$$

LIATE

$u = t \qquad du = dt$
$dv = e^t\,dt \qquad v = e^t$

$$\int u\,dv = uv - \int v\,du$$

$$= te^t - e^t + C$$
$$= \sin(x)e^{\sin(x)} - e^{\sin(x)} + C$$

We get

$$e^{\sin(x)} \cdot y = \sin(x)e^{\sin(x)} - e^{\sin(x)} + C$$

So,

$$y = e^{-\sin(x)}\left[\sin(x)e^{\sin(x)} - e^{\sin(x)} + C\right]$$

Thus,

$$y = \sin(x) - 1 + Ce^{-\sin(x)}$$

is the solution where C is any constant.