Math 2150-01

8/25/25

Topic 1 - What is a differential equation?

Ex: $\boxed{y' = 5y}$

is an example of a differential equation. The unknown is a function y.

Let's show that $y = e^{5x}$ solves this equation.

If $y = e^{5x}$, then $y' = 5e^{5x}$.

Here $y' = 5e^{5x} = 5y$.

So, $y = e^{5x}$ solves $y' = 5y$

Def:

- An equation relating an unknown function and one or more of it derivatives is called a differential equation.

- If a differential equation only has regular derivatives of a single function then it is called an ordinary differential equation or ODE.

If the equation contains partial derivatives then its called a partial differential equation or PDE.

- The order of a differential equation is the order of the highest derivative that occurs in the equation

Ex's: $y' = 3y$ ← ODE order 1

$y''' - 5y'' + y = 0$ ← ODE order 3

$\underbrace{\sin(x)\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^3}$ ← ODE order 2

Same as: $\sin(x)\,y'' + y' = x^3$

y is a function of x, so x is a number

y is a function

(Laplace's equation)

$$\underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}_{u_{xx} + u_{yy} = 0}$$

PDE
order 2

where $u = u(x,y)$ is a function of x and y.

In this class we only study ODE's. In math 4030 you study PDE's.

Def: An ODE is called <u>linear</u> if it is of the form:

$$\underbrace{a_n(x)}\, y^{(n)} + \underbrace{a_{n-1}(x)}\, y^{(n-1)} + \dots + \underbrace{a_1(x)}\, y' + \underbrace{a_0(x)}\, y = \underbrace{b(x)}$$

these coefficients only have numbers and x's in them

Recall: $y^{(k)}$ means the k-th derivative

Ex: $y^{(5)} = y'''''$

Ex:

$$\underbrace{5x}\, y'''' - \underbrace{3}\, y'' + \underbrace{\sin(x)}\, y = \underbrace{10}$$

#'s and x's

So, this is a 4th order ODE and it's linear.

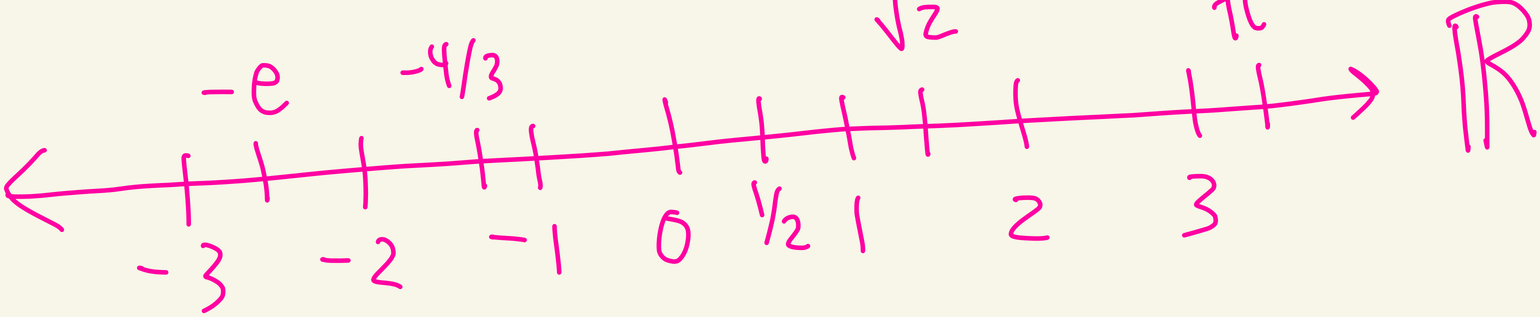
$\#$ and x's

Ex:

$$5y'' - 5\underbrace{yy'} = x$$

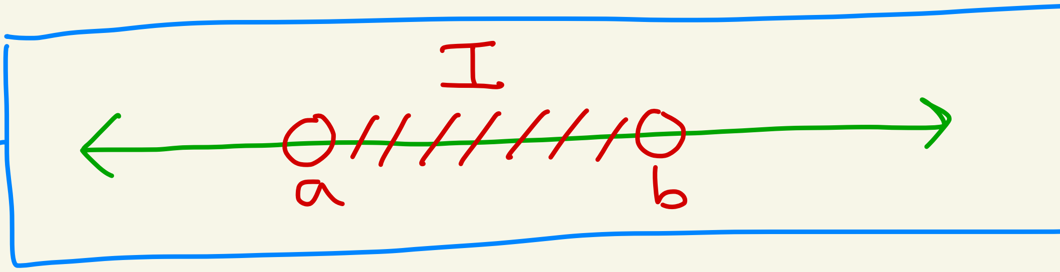
$y \cdot y'$ makes the equation not linear

Def: The set of real numbers is denoted by $\mathbb{R}$.

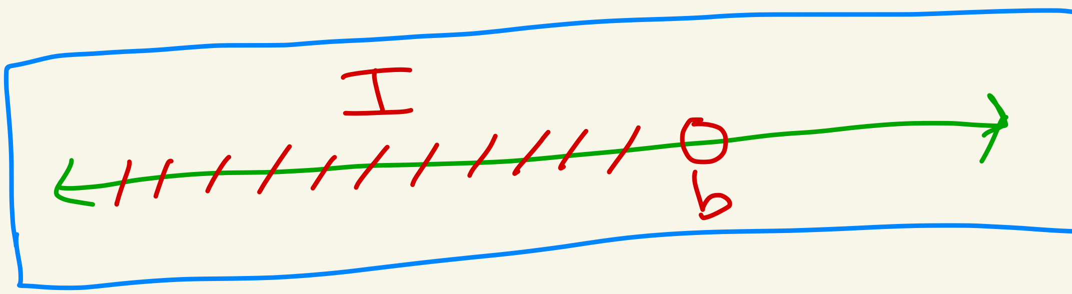


Def: An <u>open interval</u> I is an interval of one of the following forms:

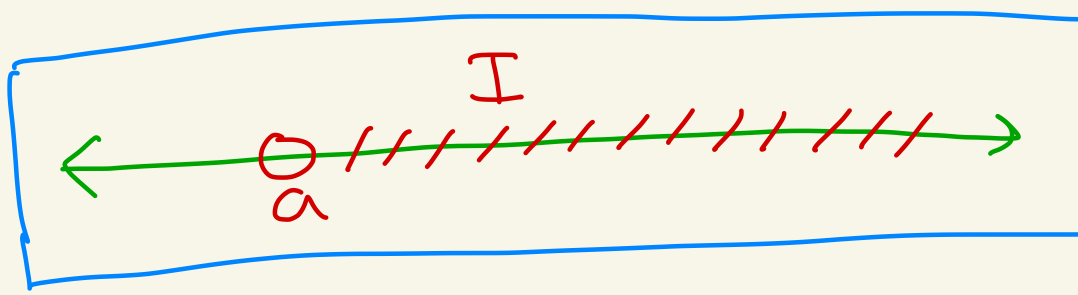
$I = (a, b)$



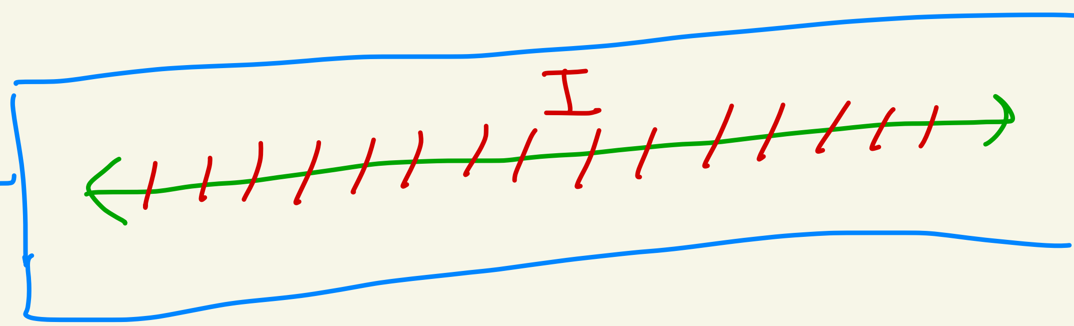
$I = (-\infty, b)$



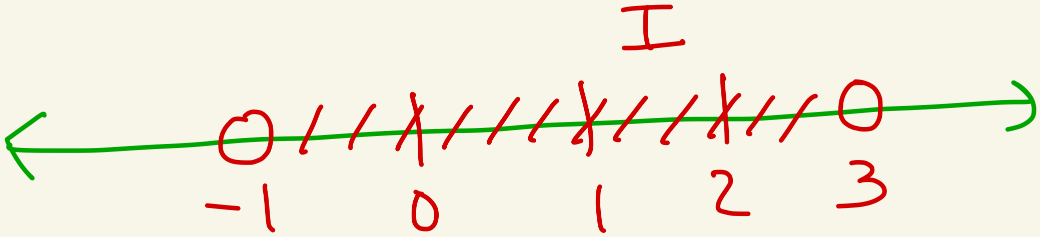
$I = (a, \infty)$



$I = (-\infty, \infty)$



Ex: $I = (-1, 3)$ is an open interval



<u>Def:</u> A function f is a <u>solution</u> to an n-th order ODE on an open interval I if:

(1) $f, f', f'', \ldots, f^{(n)}$ exist on I

and

(2) when you plug f and it's derivatives into the ODE it solves the ODE for every x in I.

In addition, sometimes one is given what

$f(x_0), f'(x_0), \ldots, f^{(n-1)}(x_0)$

must equal for some x_0 in I. This turns the ODE into an <u>initial value problem</u> (IVP).

Ex: Let's find a solution to

$y'' = -y$ on $I = (-\infty, \infty)$.

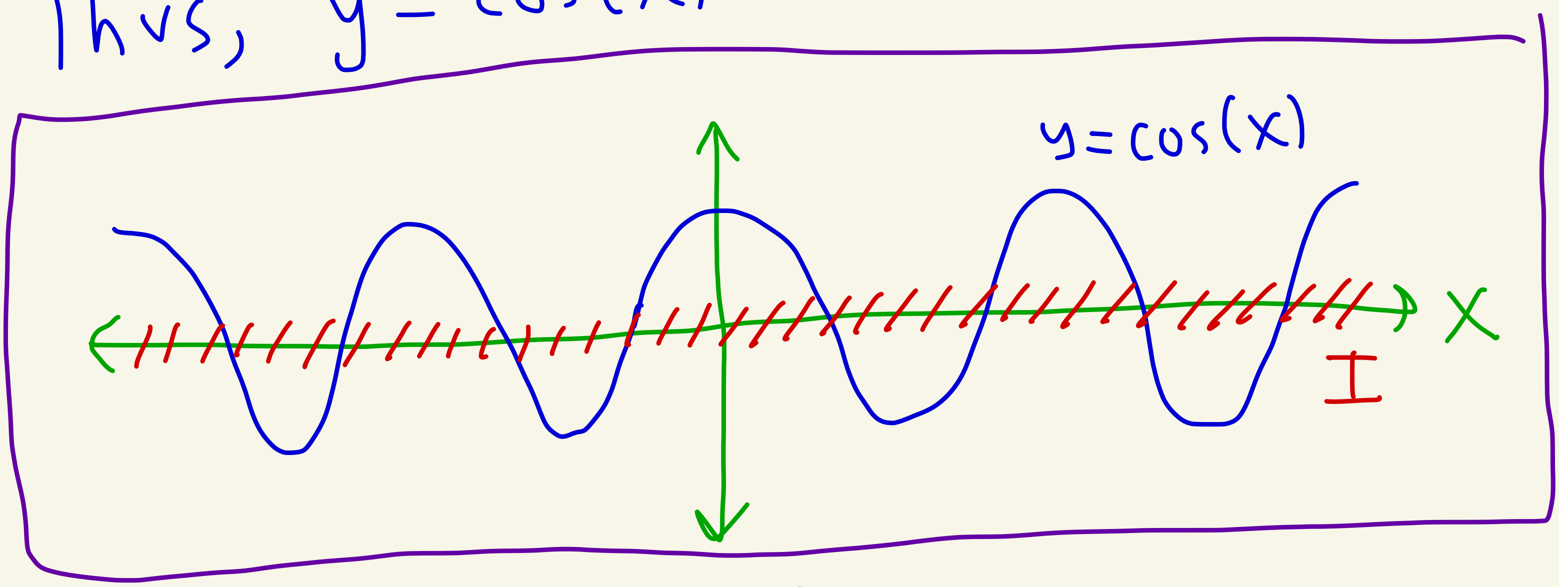
Let $y = \cos(x)$.

Then, $y' = -\sin(x)$

$y'' = -\cos(x)$.

$y'' = -y$

Thus, $y = \cos(x)$ solves $y'' = -y$.



$y = \cos(x)$ and it's derivatives are defined on $I = (-\infty, \infty)$.

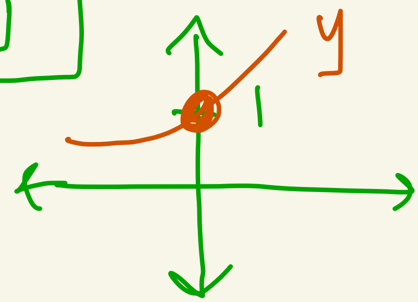
Thus, $y = \cos(x)$ solves $y'' = -y$ on I

Ex: Let's find a solution to the initial-value problem

$$\boxed{\begin{aligned} y' &= y^2 \\ y(0) &= 1 \end{aligned}}$$

non-linear ODE order 1

condition on solution



Consider $y = \frac{1}{1-x}$

Then:

$$y = (1-x)^{-1}$$

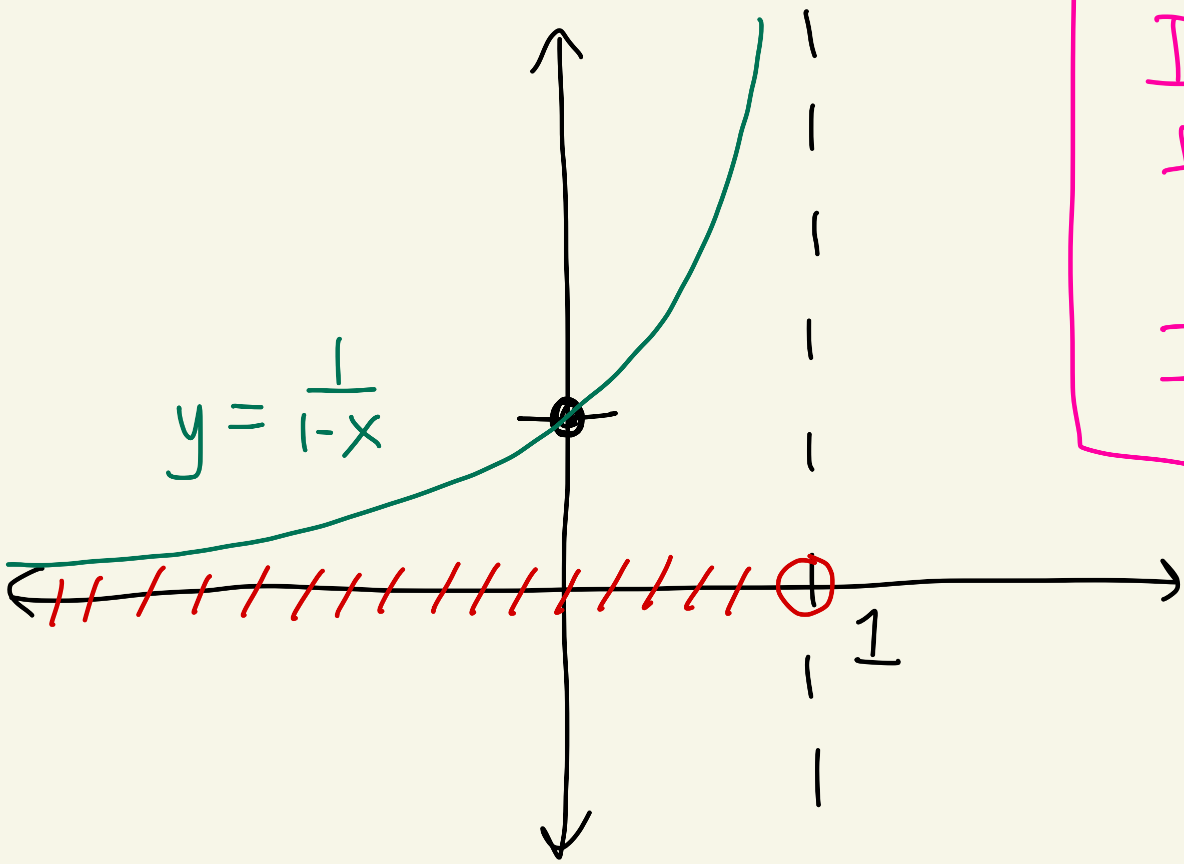
$$y' = -(1-x)^{-2}\cdot(-1) = (1-x)^{-2} = \frac{1}{(1-x)^2}$$

Then, $y' = \frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)^2 = y^2$

So, $y = \dfrac{1}{1-x}$ solves $y' = y^2$.

And $y(0) = \dfrac{1}{1-0} = 1$.

Thus, $y = \dfrac{1}{1-x}$ solve the IVP.



It solves the IVP on $I = (-\infty, 1)$