Math 2150-01

11/5/25

Ex: Let's find a power series solution to

$$\boxed{\begin{aligned} y' - 2xy &= 0 \\ y(0) &= 1 \end{aligned}} \qquad \boxed{x_0 = 0}$$

Step 1: Is there a power series solution and what is the radius of convergence?

Coefficients

$$\left.\begin{aligned} -2x &= 0 - 2x + 0x^2 + 0x^3 + \dots \\ 0 &= 0 + 0x + 0x^2 + 0x^3 + \dots \end{aligned}\right] \text{power series centered at } x_0 = 0$$

These are polynomials which are analytic everywhere including at $x_0 = 0$. Both of the above series have radius of convergence $r = \infty$.

So we will have a power series solution of the form:

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y''''(0)}{4!}x^4 + \dots$$

will radius of convergence $r = \infty$

Step 2: Find the first few terms of the power series.

Given: $\boxed{\begin{array}{l} y(0) = 1 \\ y' - 2xy = 0 \end{array}}$

$\boxed{y(0) = 1}$

$\boxed{y' = 2xy}$

$$\boxed{\begin{aligned} y'(0) &= (2)(0)[y(0)] \\ &= (2)(0)(1) \\ &= 0 \end{aligned}}$$

Next find $y''(0)$.

Differentiate $y' = 2xy$ to get

$$y'' = (2)y + (2x)y'$$

$$y'' = 2y + 2xy'$$

plug in $x=0$

$$y''(0) = 2[y(0)] + 2(0)[y'(0)]$$

$$= 2[1] + 2(0)[0]$$

$$= 2$$

$$\boxed{y''(0) = 2}$$

Now differentiate $y'' = 2y + 2xy'$ to get:

$$y''' = 2y' + (2y' + 2xy'')$$

$$y''' = 4y' + 2xy''$$

$$y'''(0) = 4[y'(0)] + 2(0)[y''(0)]$$

$$= 4[0] + 2(0)[2]$$

$$= 0$$

$$\boxed{y'''(0) = 0}$$

Differentiate $y''' = 4y' + 2xy''$ to get:

$$y'''' = 4y'' + (2y'' + 2xy''')$$

$$y'''' = 6y'' + 2xy'''$$

$$y''''(0) = 6[y''(0)] + 2(0)[y'''(0)]$$

$$= 6(2) + 0$$

$$\boxed{y''''(0) = 12}$$

So we have:

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y''''(0)}{4!}x^4 + \cdots$$

$$= 1 + \underbrace{\frac{0}{1!}}_{0}x + \underbrace{\frac{2}{2!}}_{1}x^2 + \underbrace{\frac{0}{3!}}_{0}x^3 + \underbrace{\frac{12}{4!}}_{1/2}x^4 + \cdots$$

$$= \boxed{1 + x^2 + \frac{1}{2}x^4 + \cdots}$$

with radius of convergence $r = \infty$

[Topic 3 method gives $y = e^{x^2}$]

Ex: Consider

$$\boxed{\begin{gathered} y'' + x^2 y' - (x-1)y = \ln(x) \\ y'(1) = 0 \\ y(1) = 0 \end{gathered}}$$

$\boxed{x_0 = 1}$

Are the coefficients analytic at $x_0 = 1$?

coefficients

$$\left.\begin{aligned} x^2 &= 1 + 2(x-1) + (x-1)^2 + 0(x-1)^3 + \cdots \\ -(x-1) &= 0 - 1\cdot(x-1) + 0(x-1)^2 + 0(x-1)^3 + \cdots \end{aligned}\right] r = \infty$$

$$\ln(x) = (x-1) + \tfrac{1}{2}(x-1)^2 + \cdots \;\Big]\; r = 1$$

The min r from above is $r = 1$.

Thus, we will have a power

Series solution with radius of convergence at least $r=1$.

Let's find y.

Given:

$$\boxed{\begin{aligned} y(1) &= 0 \\ y'(1) &= 0 \end{aligned}}$$

$$y'' + x^2 y' - (x-1)y = \ln(x)$$

$$y'' = \ln(x) - x^2 y' + (x-1)y$$

$$y''(1) = \ln(1) - (1)^2 [y'(1)] + (1-1)[y(1)]$$

$$= 0 - (1)[0] + (0)[0]$$

$$= 0$$

$$y''(1) = 0$$

Differentiate $y'' = \ln(x) - x^2 y' + (x-1)y$ to get

$$y''' = \frac{1}{x} - 2xy' - x^2 y'' + 1 \cdot y + (x-1)y'$$

$$y''' = \frac{1}{x} + y + (-x-1)\,y' - x^2 y''$$

$$y'''(1) = \frac{1}{1} + \underbrace{y(1)}_{0} + (-1-1)\underbrace{[y'(1)]}_{0} - (1)^2 \underbrace{[y''(1)]}_{0}$$

$$\boxed{y'''(1) = 1}$$

As above you can calculate

$$\boxed{y''''(1) = -3}$$

Thus,

$$y(x) = \overbrace{y(1)}^{0} + \frac{\overbrace{y'(1)}^{0}}{1!}(x-1) + \frac{\overbrace{y''(1)}^{0}}{2!}(x-1)^2$$

$$+ \frac{\overbrace{y'''(1)}^{1}}{3!}(x-1)^3 + \frac{\overbrace{y''''(1)}^{-3}}{4!}(x-1)^4 + \cdots$$

$$= \frac{1}{3!}(x-1)^3 + \frac{-3}{4!}(x-1)^4 + \cdots$$

$$= \boxed{\frac{1}{6}(x-1)^3 - \frac{1}{8}(x-1)^4 + \cdots}$$

radius of convergence at least $r=1$.