Math 2150-01

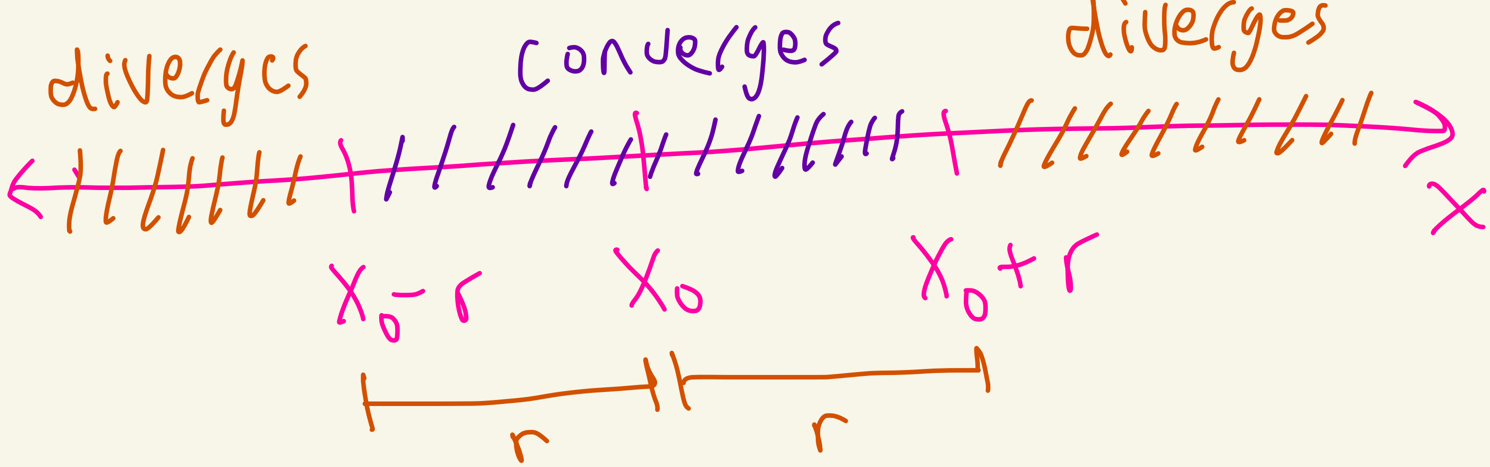
11/3/25

Topic 12 - Power series solutions to ODEs

<u>Def:</u> We say that a function $f(x)$ is <u>analytic at x_0</u> if it has a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

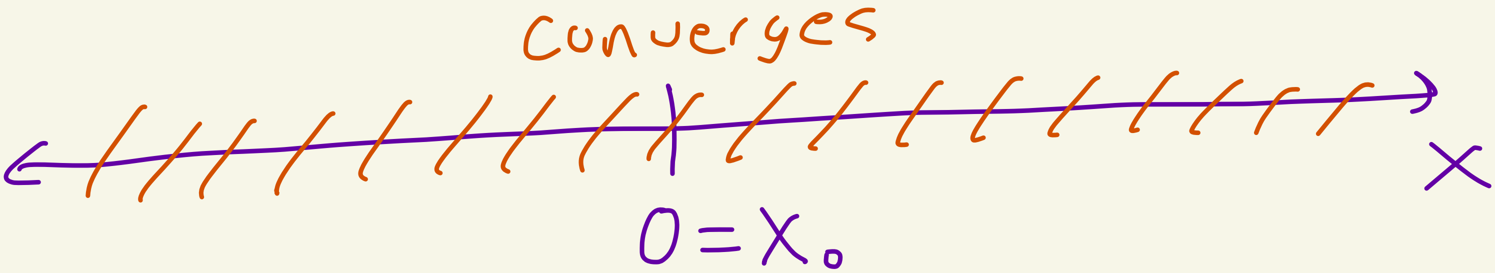
centered at x_0 with positive radius of convergence $r > 0$
[$r = \infty$ is allowed]



Ex: $x_0 = 0$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

radius of convergence $r = \infty$

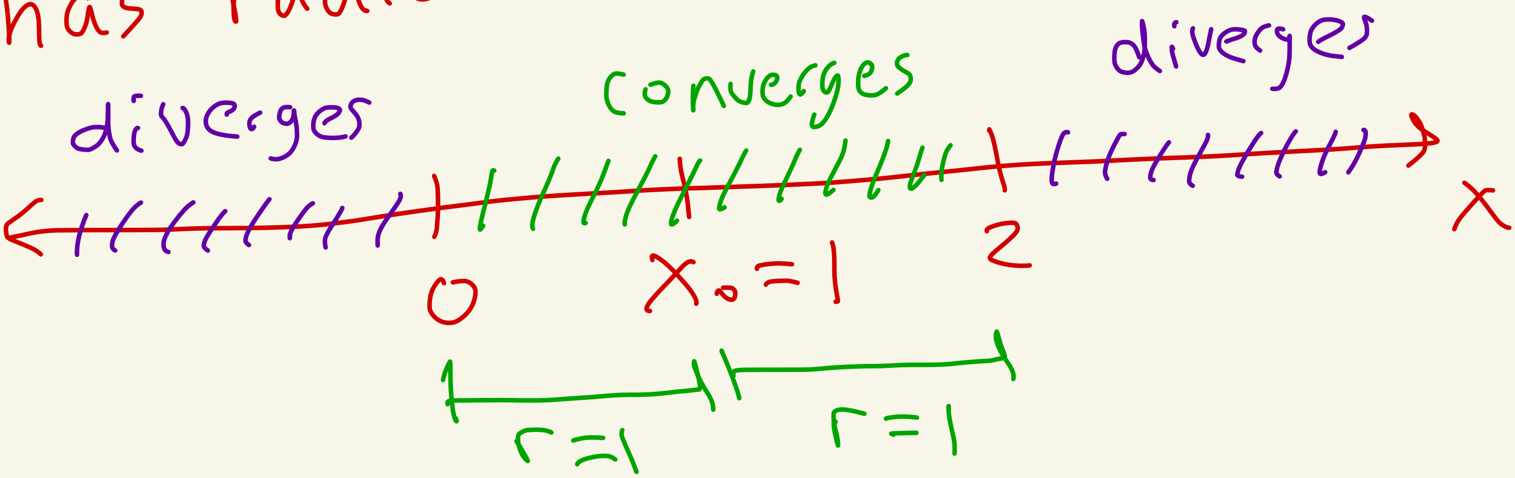


So, $\sin(x)$ is analytic at $x_0 = 0$

Ex: $x_0 = 1$

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$

has radius of convergence $r = 1$

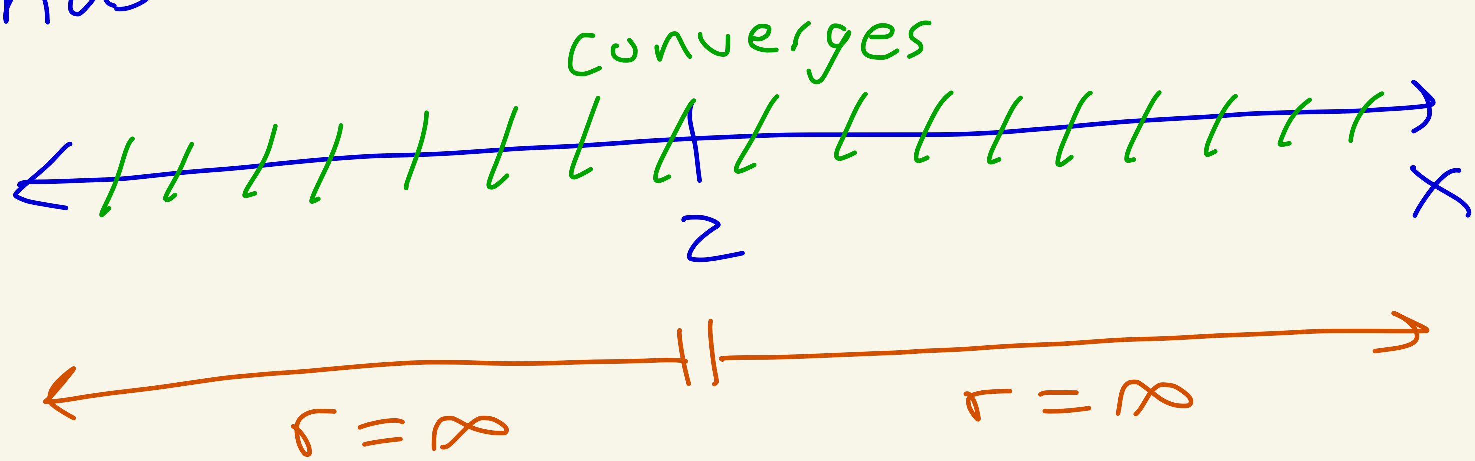


So, $\frac{1}{x}$ is analytic at $x_0 = 1$.

Ex: $x_0 = 2$

$$x^2 = 4 + 4(x-2) + (x-2)^2$$

has radius of convergence $r = \infty$



So, x^2 is analytic at $x_0 = 2$

Facts:

- polynomials are analytic at every x_0
- e^x, $\sin(x)$, $\cos(x)$ are analytic at every x_0
- rational functions (ratio of polynomials) are analytic at all x_0 except possibly when the denominator is zero

<u>Ex:</u> $x^2 + 5x + 3$ polynomial

is analytic at every x_0

Ex: $\dfrac{x}{x^2-1}$ ← rational function

is analytic for all x_0
except when $\underbrace{x_0 = \pm 1}$

$x^2 - 1 = 0$
$x = \pm 1$

For example, consider $x_0 = 0$.

$$\frac{x}{x^2-1} = \frac{x}{-1}\left[\frac{1}{1-x^2}\right]$$

$$= -x\left[1 + (x^2) + (x^2)^2 + (x^2)^3 + \cdots\right]$$

$1 + u + u^2 + u^3 + \cdots = \dfrac{1}{1-u}$
when $|u| < 1$

$u = x^2$ and $|x^2| < 1 \rightarrow |x|^2 < 1 \rightarrow |x| < 1$

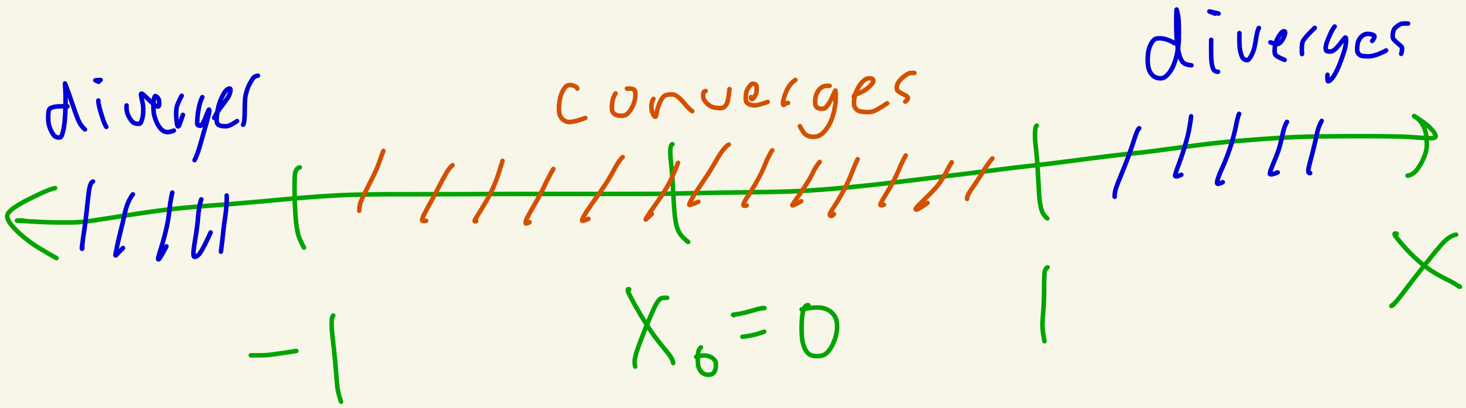
$$= -x\left[1 + x^2 + x^4 + x^6 + \dots\right]$$

$$= -x - x^3 - x^5 - x^7 - \dots$$

So,

$$\frac{x}{x^2-1} = -x - x^3 - x^5 - x^7 - \dots$$

when $-1 < x < 1$



Main Theorem

Consider either of the initial-value problems:

$$\boxed{\begin{aligned} y' + a_0(x)y &= b(x) \\ y(x_0) &= y_0 \end{aligned}}$$

first order

OR

$$\boxed{\begin{aligned} y'' + a_1(x)y' + a_0(x)y &= b(x) \\ y'(x_0) &= y_0' \\ y(x_0) &= y_0 \end{aligned}}$$

second order

In either case, if the $a_i(x)$ and $b(x)$ are all analytic at x_0 then there exists a unique solution to the initial-value problem of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

centered at x_0.

Furthermore, the radius of convergence $r > 0$ for the power series of the solution $y(x)$ is at least the smallest radius of convergence from amongst the power series of the $a_i(x)$ and $b(x)$.

Ex: Suppose $a_1(x)$ is analytic at x_0 with radius of convergence $r = 2$ and $b(x)$ is analytic at x_0

with radius of convergence
$r = 5$.

Then,

$$\boxed{\begin{gathered} y' + a_1(x)\, y = b(x) \\ y(x_0) = y_0 \end{gathered}}$$

will have a solution

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with radius of convergence
at least $\underbrace{r = 2}_{\text{minimum of 2 and 5 from above}}$