Math 2150-01
10/6/25

HW 3

② (b) Solve the linear equation

$$x^2 y' + x(x+2)y = e^x$$

on $I = (0, \infty)$. ← $\boxed{x > 0}$

Divide by x^2 to get

$$y' + \frac{x(x+2)}{x^2} y = \frac{1}{x^2} e^x$$

$$\frac{x(x+2)}{x^2} = \frac{(x+2)}{x} = 1 + \frac{2}{x}$$

Simplify

$$y' + \left(1 + \frac{2}{x}\right) y = \frac{1}{x^2} e^x$$

Let

$$A(x) = \int \left(1 + \frac{2}{x}\right) dx = x + 2\ln|x|$$

$$= x + 2\ln(x)$$

$x > 0$

We have

$$e^{A(x)} = e^{x + 2\ln(x)} = e^{x} e^{2\ln(x)}$$

$$= e^{x} e^{\ln(x^2)}$$

$A\ln(B) = \ln(B^A)$

$$= e^{x} \cdot x^2 = x^2 e^{x}$$

$e^{\ln(C)} = C$

Multiply

$$y' + \left(1 + \frac{2}{x}\right) y = \frac{1}{x^2} e^x$$

by $e^{A(x)} = x^2 e^x$ to get:

$$\underbrace{x^2 e^x y' + x^2 e^x \left(1 + \frac{2}{x}\right) y}_{\left(e^{A(x)} \cdot y\right)'} = \underbrace{x^2 e^x \left(\frac{1}{x^2} e^x\right)}_{e^{2x}}$$

So we get

$$\left(x^2 e^x y\right)' = e^{2x}$$

Integrate to get

$$x^2 e^x y = \int e^{2x} \, dx$$

$$x^2 e^x y = \frac{1}{2} e^{2x} + C$$

$$\boxed{y = \frac{1}{x^2 e^x} \left(\frac{1}{2} e^{2x} + C \right)}$$

$$\boxed{y = \frac{1}{2} \frac{1}{x^2} e^x + C \frac{1}{x^2 e^x}}$$

$\boxed{\text{HW 4}}$

① (c) Solve the separable IVP:

$$\frac{dy}{dx} = \frac{-x}{y}, \quad y(4) = 3$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

So,

$$y\,dy = -x\,dx$$

Then

$$\int y\,dy = -\int x\,dx$$

So,

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$\frac{y^2}{2} + C_1 = -\frac{x^2}{2} + C_2$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + \underbrace{(C_2 - C_1)}_{C}$$

Then,

$$y^2 = -x^2 + 2C$$

Let's find C.

We have $y(4) = 3$.

Plug in $x = 4, y = 3$ to get

$$(3)^2 = -(4)^2 + 2C$$

So,

$$9 = -16 + 2C$$

$$25 = 2C$$

$$\frac{25}{2} = C$$

Plug this back in to $y^2 = -x^2 + 2C$ to get

$$y^2 = -x^2 + 2\left(\frac{25}{2}\right)$$

So,

$$y^2 = -x^2 + 25$$

Thus,

$$y = \pm\sqrt{-x^2 + 25}$$

To decide if $+$ or $-$ use $y(4) = 3$.

We want $\underbrace{\pm\sqrt{-(4)^2 + 25}}_{y(4)} = 3$

We want $\pm\sqrt{9} = 3$

Need + not $-$.

So,

$$\boxed{y = \sqrt{-x^2+25}}$$

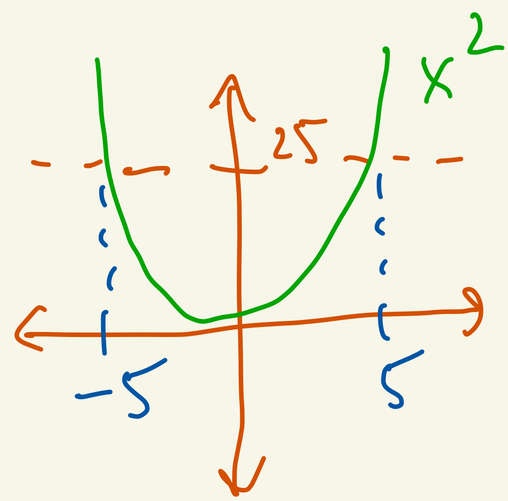
Where is this function defined?

When:

$$-x^2 + 25 \geq 0$$

$$25 \geq x^2$$

$$-5 \leq x \leq 5$$



$\boxed{\text{Hw 5}}$

① (d) $$\underbrace{\frac{2x}{y}}_{M = 2xy^{-1}} - \underbrace{\frac{x^2}{y^2}}_{N = -x^2y^{-2}} \frac{dy}{dx} = 0$$

Test if exact

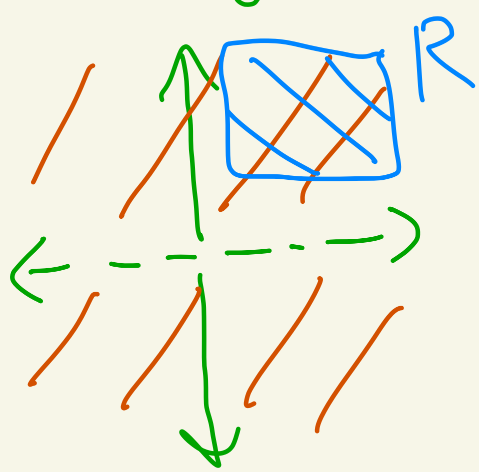
$M = 2xy^{-1}$

$N = -x^2y^{-2}$

$$\boxed{\begin{aligned} \frac{\partial M}{\partial y} &= -2xy^{-2} \\ \frac{\partial N}{\partial x} &= -2xy^{-2} \end{aligned}} \quad \text{equal} \quad \checkmark$$

$$\frac{\partial M}{\partial x} = 2y^{-1}, \quad \frac{\partial N}{\partial y} = 2x^2y^{-3}$$

all defined when $y \neq 0$



The equation is exact.

Let's solve it. We solve:

$$\boxed{\begin{aligned}\frac{\partial f}{\partial x} &= M \\ \frac{\partial f}{\partial y} &= N\end{aligned}} \longrightarrow \boxed{\begin{aligned}\frac{\partial f}{\partial x} &= 2xy^{-1} \quad (1) \\ \frac{\partial f}{\partial y} &= -x^2y^{-2} \quad (2)\end{aligned}}$$

Integrate (1) with respect to x:

$$f(x,y) = x^2y^{-1} + \underbrace{C(y)}_{\text{constant with respect to } x}$$

Integrate (2) with respect to y:

$$f(x,y) = -x^2\frac{y^{-1}}{-1} + \underbrace{D(x)}_{\text{constant with respect to } y}$$

Set equal to get

$$\cancel{x^2y^{-1}} + C(y) = \cancel{x^2y^{-1}} + D(x)$$

So,

$$C(y) = D(x)$$

Set $C(y)=0$, $D(x)=0$.

Thus,

$$f(x,y) = x^2y^{-1} + D(x)$$
$$= x^2y^{-1} + 0$$
$$= x^2y^{-1}$$

Answer: $\boxed{x^2y^{-1} = c}$

$\boxed{\frac{1}{c}x^2 = y}$

c is a constant

$\boxed{\text{HW 6}}$

(2) Consider

$$x^2 y'' - 5xy' + 8y = 24$$

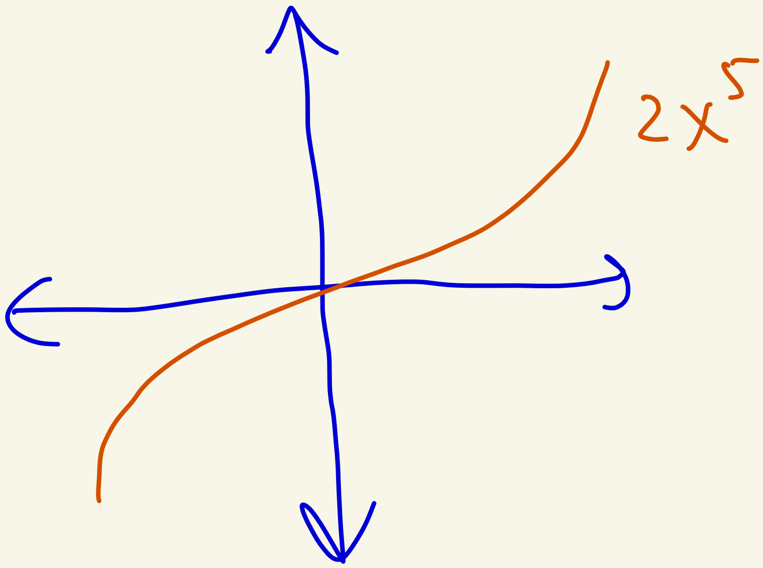
on $I = (-\infty, \infty)$.

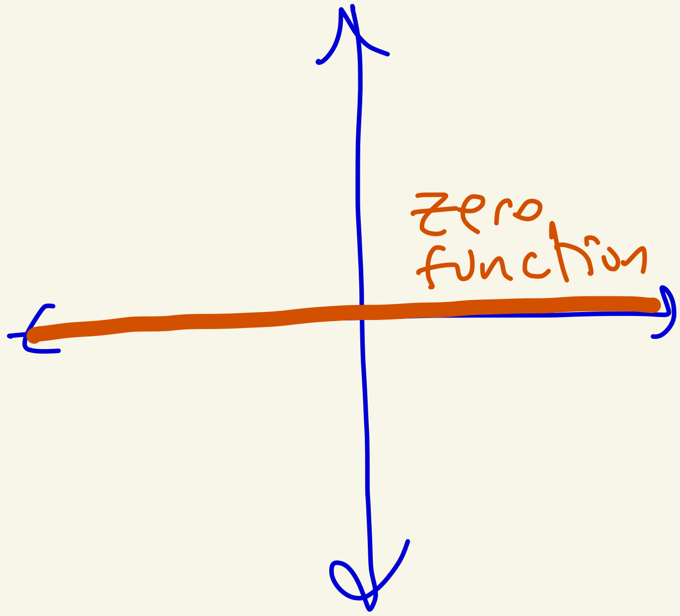
(a) Show that $y_1 = x^2$, $y_2 = x^4$ are linearly independent on I.

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^4 \\ 2x & 4x^3 \end{vmatrix} = (x^2)(4x^3) - (x^4)(2x)$$

$$= 2x^5$$

This is not the zero function on I.





For example at $x=1$, the Wronskian is $2x^5 = 2(1)^5 \neq 0$.

Thus, $y_1 = x^2$, $y_2 = x^4$ are lin. ind. on I.

Now show $y_1 = x^2$, $y_2 = x^4$ both solve

$$x^2 y'' - 5xy' + 8y = 0$$

For $y_1 = x^2$ we get: $\boxed{y_1' = 2x,\ y_1'' = 2}$

$$x^2\underbrace{(2)}_{y_1''} - 5x\underbrace{(2x)}_{y_1'} + 8\underbrace{(x^2)}_{y_1}$$

$$= 2x^2 - 10x^2 + 8x^2$$

$$= 0$$

So, y_1 solves the eqn.

For $y_2 = x^4$ we get: $\boxed{y_2' = 4x^3,\ y_2'' = 12x^2}$

$$x^2\underbrace{(12x^2)}_{y_2''} - 5x\underbrace{(4x^3)}_{y_2'} + 8\underbrace{(x^4)}_{y_2}$$

$$= 12x^4 - 20x^4 - 8x^4$$

$$= 0$$

So, y_2 solves the eqn.

(c) What's the general solution to $x^2y'' - 5xy' + 8y = 0$?

$$\boxed{\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 x^2 + c_2 x^4 \end{aligned}}$$

$\boxed{\text{HW 7}}$ 1(e)

Solve

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$$

$$y'' + 8y' + 16y = 0$$

$$r^2 + 8r + 16 = 0$$

$$\underbrace{(r+4)}_{\substack{r+4=0 \\ r=-4}}\underbrace{(r+4)}_{\substack{r+4=0 \\ r=-4}} = 0$$

Repeated real root $r = -4$

Answer:

$$y_h = c_1 e^{-4x} + c_2 x e^{-4x}$$