Math 2150-01
10/29/25

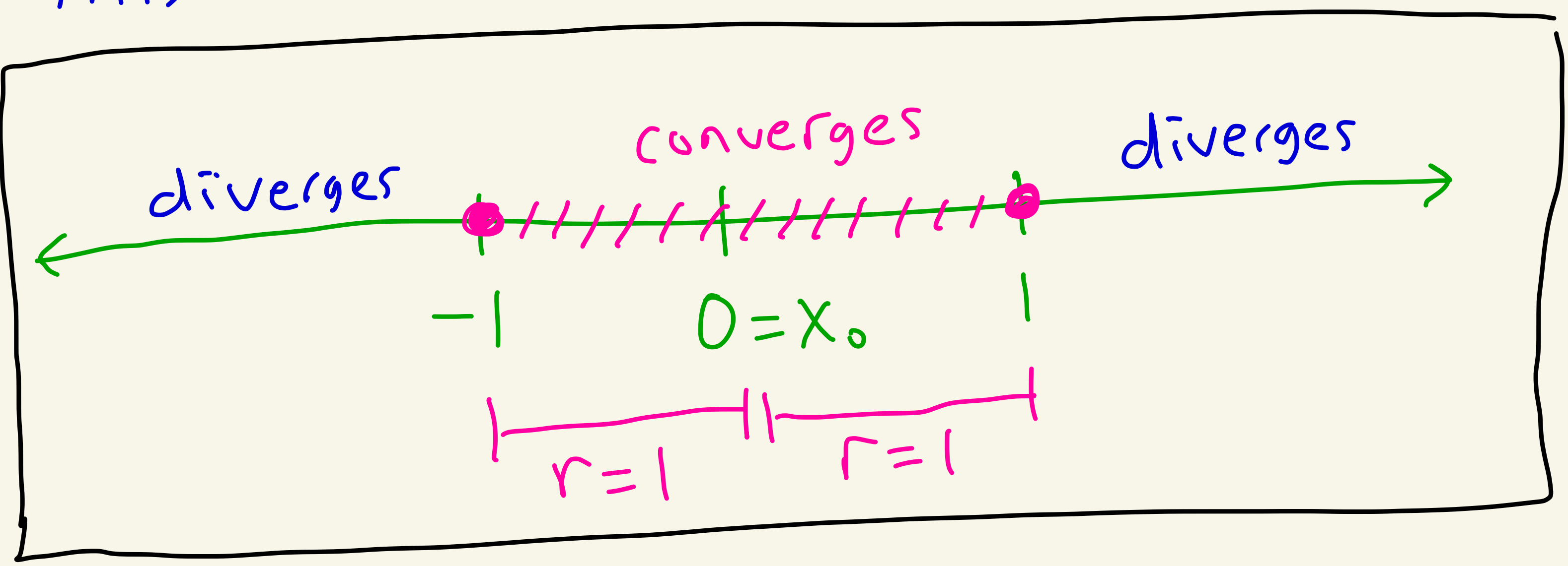
Ex: From Calc II we have

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

centered at $x_0 = 0$.

This series converges for $-1 \leq x \leq 1$.



radius of convergence is $r = 1$

Side note: You can use this series to approximate π.

Plug $x = 1$ into the series to get

$$\frac{\pi}{4} = \tan^{-1}(1) = 1 - \frac{1}{3}(1)^3 + \frac{1}{5}(1)^5 - \frac{1}{7}(1)^7 + \cdots$$

So,

$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

Taylor series

If

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

has radius of convergence $r > 0$,

then

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

$f^{(n)}$ is the n-th derivative of f

So,

$$f(x) = \underbrace{\frac{f^{(0)}(x_0)}{0!}}_{a_0} + \underbrace{\frac{f'(x_0)}{1!}}_{a_1}(x - x_0) + \underbrace{\frac{f''(x_0)}{2!}}_{a_2}(x - x_0)^2$$

$$+ \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$$
$$+ \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

<u>Ex:</u> Find the power series for $f(x) = x^2$ centered at $x_0 = 2$.

Right now $f(x) = x^2$ is already a power series centered at $x_0 = 0$. It is:

$$x^2 = 0 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4 + \cdots$$

Let's now center it at $x_0 = 2$

$f(x) = x^2 \longrightarrow f(2) = 2^2 = 4$

$f'(x) = 2x \longrightarrow f'(2) = 2(2) = 4$

$f''(x) = 2 \longrightarrow f''(2) = 2$

$\left.\begin{array}{l} f'''(x) = 0 \\ f^{(n)}(x) = 0 \end{array}\right] \longrightarrow f^{(n)}(2) = 0, \ n \geq 3$

$f^{(4)}(x) = 0$ $\Big\downarrow$ for $n \geq 3$

$\vdots$

So we get:

$$x^2 = f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^{(4)}(2)}{4!}(x-2)^4 + \cdots$$

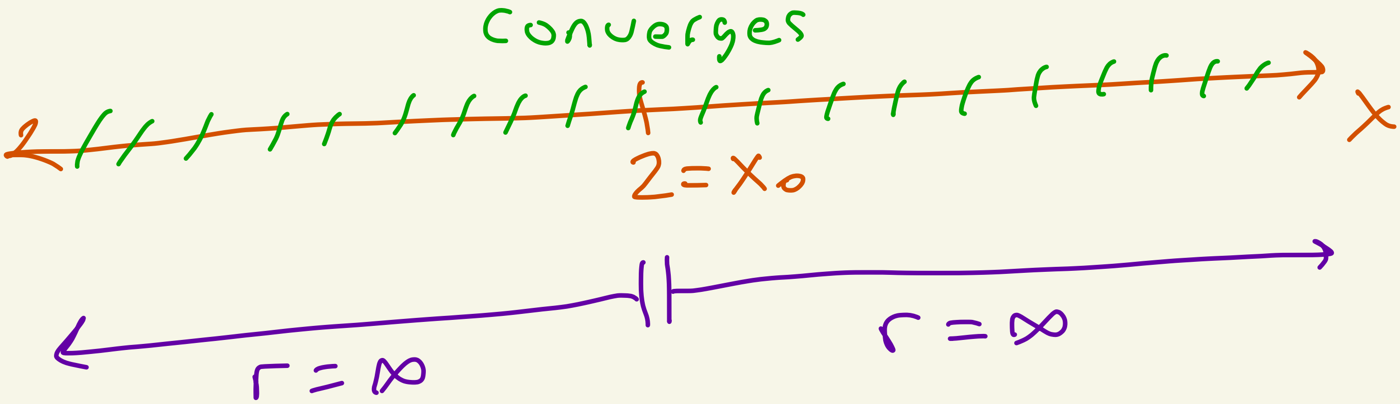
$$= \boxed{4 + 4(x-2) + \frac{2}{2!}(x-2)^2} + \underbrace{\frac{0}{3!}(x-2)^3 + \frac{0}{4!}(x-2)^4 + \cdots}_{\text{all zero}}$$

So,

$$\boxed{x^2 = f(x) = 4 + 4(x-2) + (x-2)^2}$$

What is the radius of convergence?

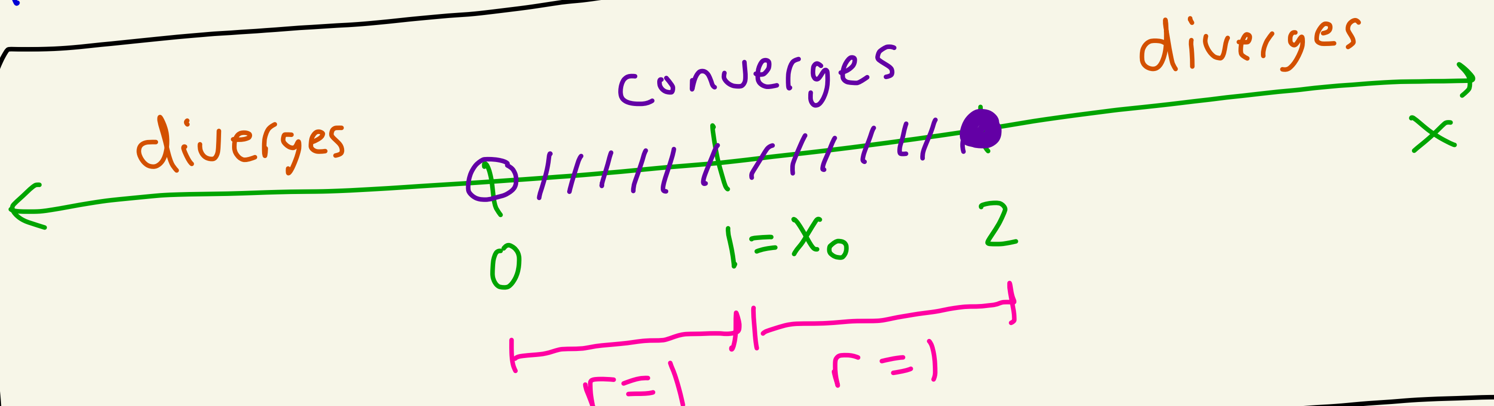
It converges for all x. The radius of convergence is $r = \infty$



Fact: You can differentiate or integrate a function by differentiating or integrating it's power/Taylor series term by term. This process won't change the radius of convergence.

Ex: Last time we recalled that

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$



That is, it converges for $0 < x \le 2$.

Differentiate both sides to get:

$$\frac{1}{x} = 1 - \frac{1}{2}\left[2(x-1)^1\right] + \frac{1}{3}\left[3(x-1)^2\right] - \frac{1}{4}\left[4(x-1)^3\right] + \dots$$

$$\boxed{\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots}$$

This will still have radius of convergence $r = 1$, but convergence changes at the endpoints.

The $1/x$ series above only converges for $0 < x < 2$.