Math 2150-01

10/27/25

(Topic 11 continued...)

Def: A <u>power series</u> is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \underbrace{a_0}_{\substack{n=0 \\ \text{term}}} + \underbrace{a_1(x-x_0)}_{n=1 \text{ term}}$$

$$+ \underbrace{a_2(x-x_0)^2}_{n=2 \text{ term}} + \underbrace{a_3(x-x_0)^3}_{n=3 \text{ term}}$$

$$+ \underbrace{a_4(x-x_0)^4}_{n=4 \text{ term}} + \cdots$$

where $a_0, a_1, a_2, a_3, \ldots$ are constants x_0 is a constant (called the <u>center</u> of the power series), x is a variable

Ex: (Geometric series)

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$= \underbrace{1}_{a_0} + \underbrace{1}_{a_1} \cdot (x - \underset{x_0=0}{0}) + \underbrace{1}_{a_2} \cdot (x - \underset{x_0=0}{0})^2 + \cdots$$

This is a power series centered at $x_0 = 0$.

In Calculus II you learn that the above series converges when $-1 < x < 1$ and diverges otherwise.

And you learn that if $-1 < x < 1$, then

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

$\boxed{-1 < x < 1}$

If you plug in $x = \frac{1}{5}$ then

$$\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = 1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \dots$$

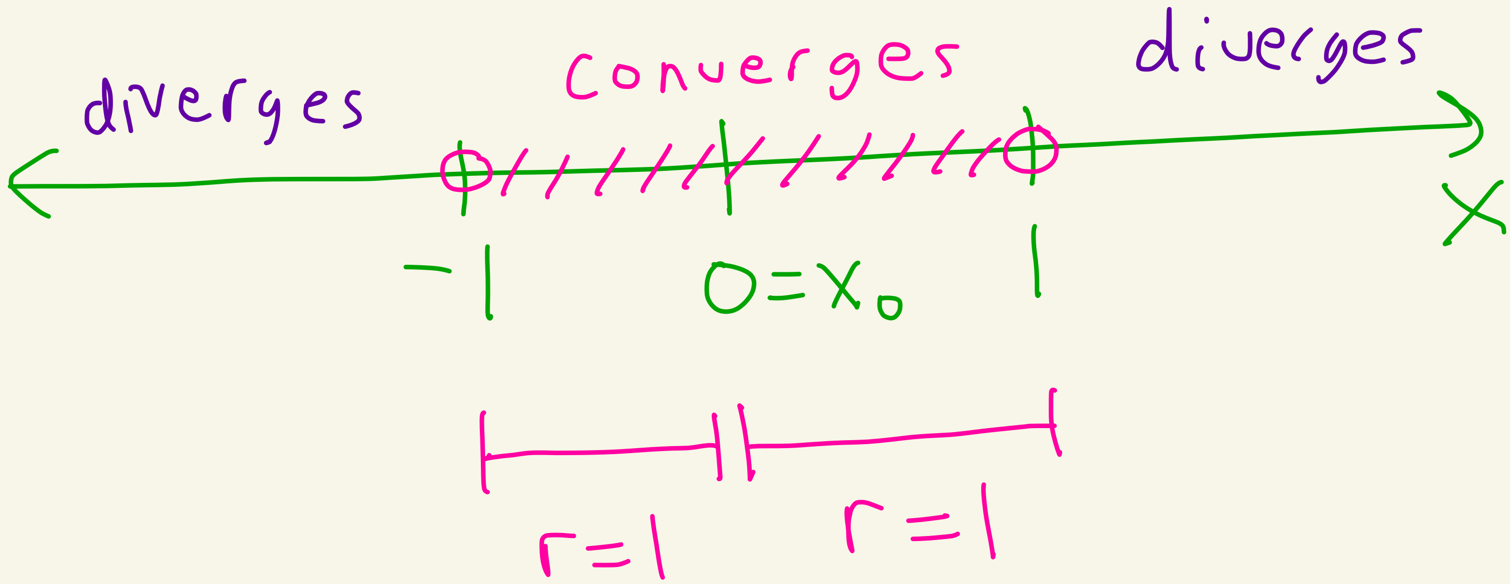
$$= \frac{1}{1 - \frac{1}{5}} = \boxed{\frac{5}{4}}$$

If $x = 2$, then

$$\sum_{n=0}^{\infty} 2^n = 1 + 2 + 2^2 + 2^3 + 2^4 + \dots$$

diverges.

Picture for where $\sum_{n=0}^{\infty} x^n$ converges:



$r=1$ is called the radius of convergence of $\sum_{n=0}^{\infty} x^n$

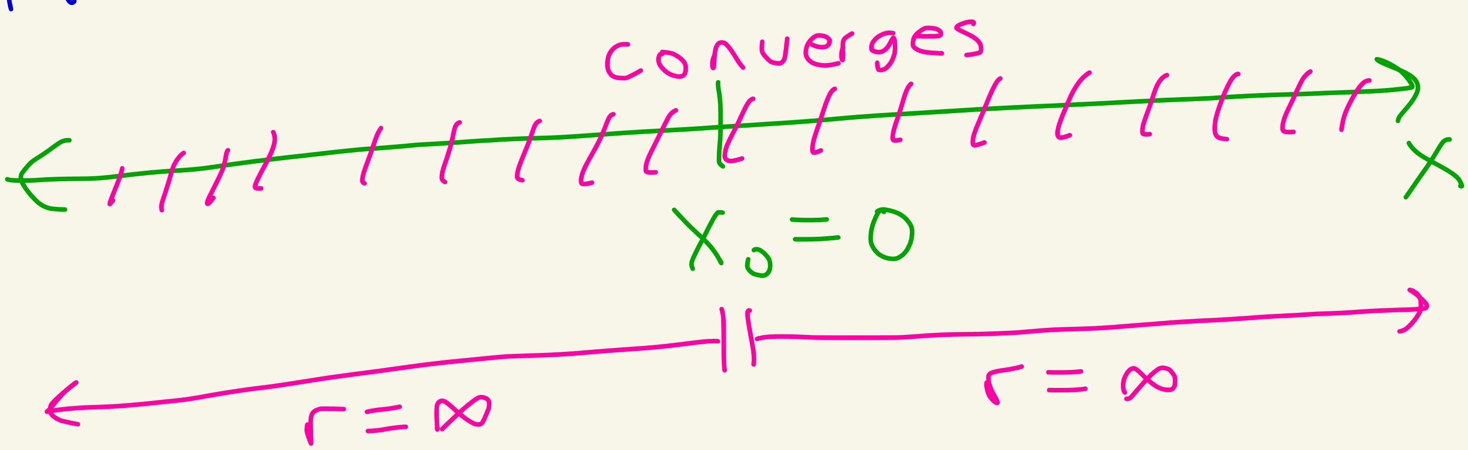
Ex: In Calc II you learn that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

$\boxed{\begin{array}{l} 0! \\ 1! = 1 \\ 2! = 2\cdot 1 = 2 \\ 3! = 3\cdot 2\cdot 1 = 6 \\ 4! = 4\cdot 3\cdot 2\cdot 1 \\ \quad = 24 \end{array}}$

This is centered at $x_0 = 0$.
The series converges for all x.



radius of convergence is $r = \infty$

For example,

$$e = e^1 = 1 + 1 + \frac{1}{2!}1^2 + \frac{1}{3!}1^3 + \cdots$$

$$= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

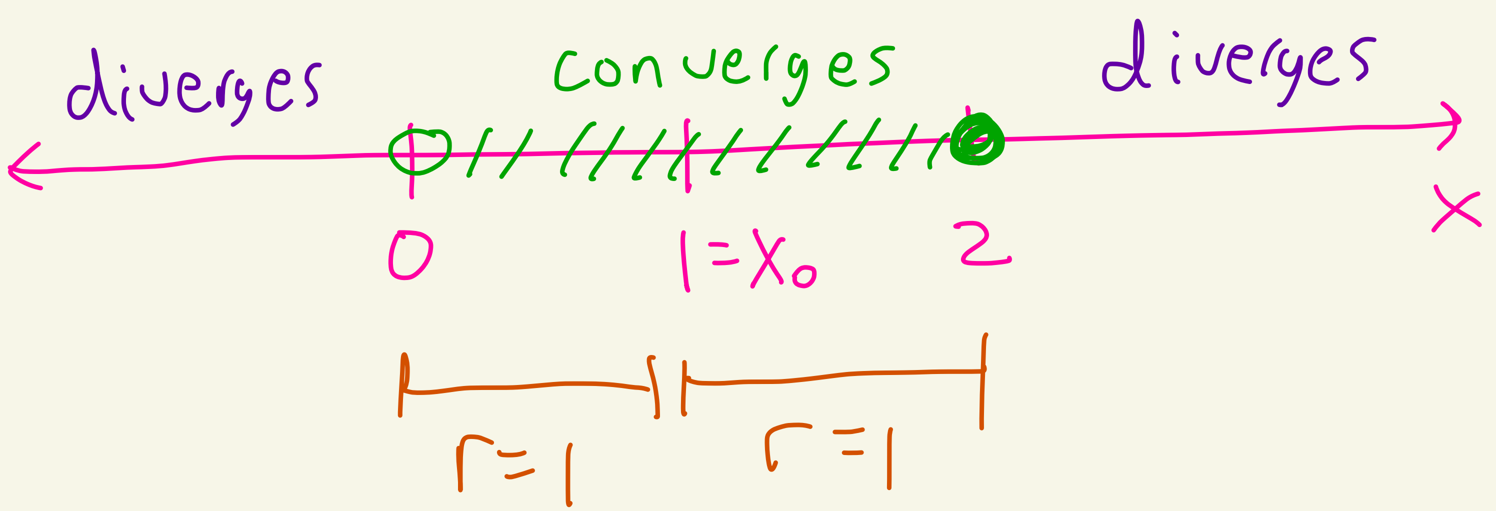
Ex: In Calc II, you learn that

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n \qquad \boxed{0 < x \leq 2}$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \cdots$$

This series is centered at $x_0 = 1$

And it converges when $0 < x \leq 2$



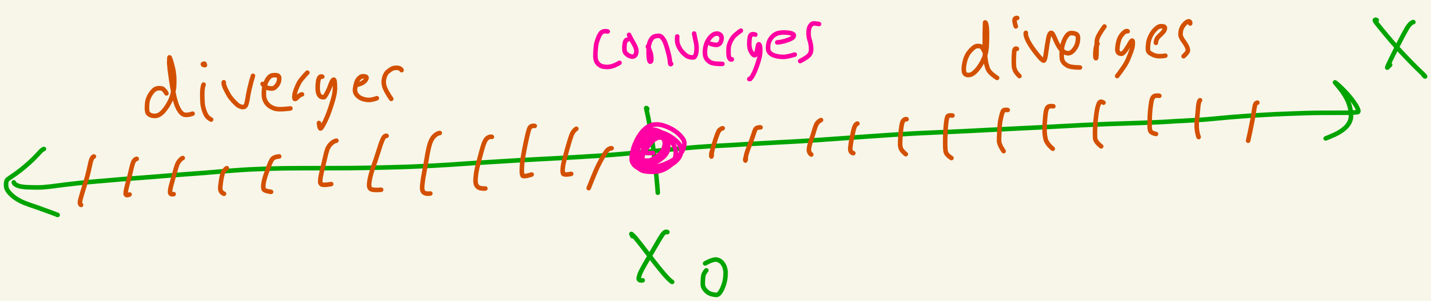
In general, consider the power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

centered at x_0.

There are 3 possibilities:

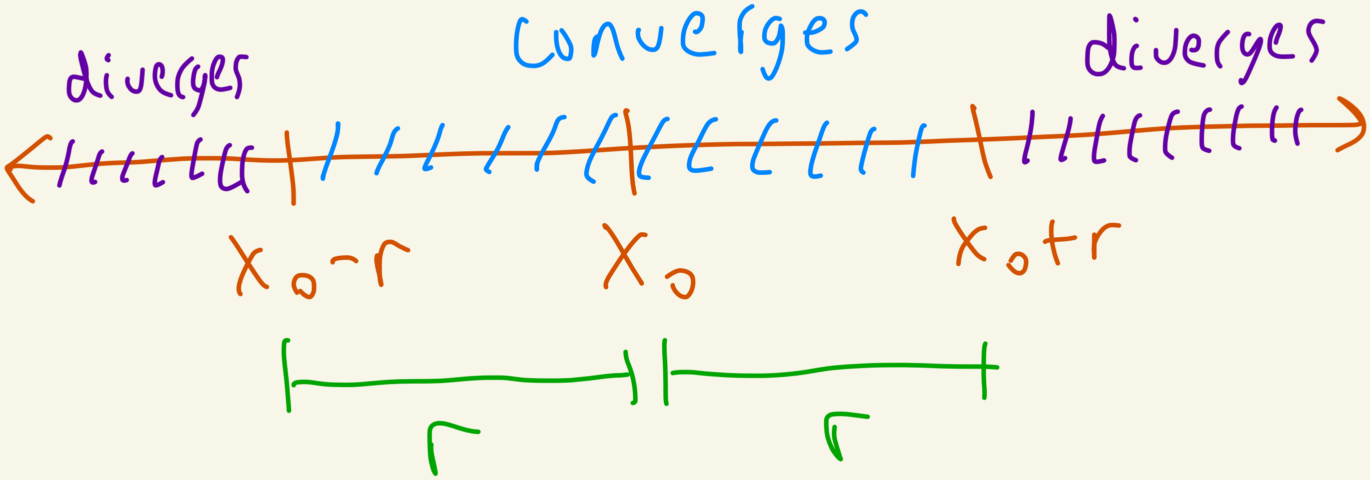
① The series only converges when $x = x_0$



$r = 0$ is the radius of convergence

② there exists $r>0$ where the series convergences for

$$x_0 - r < x < x_0 + r$$

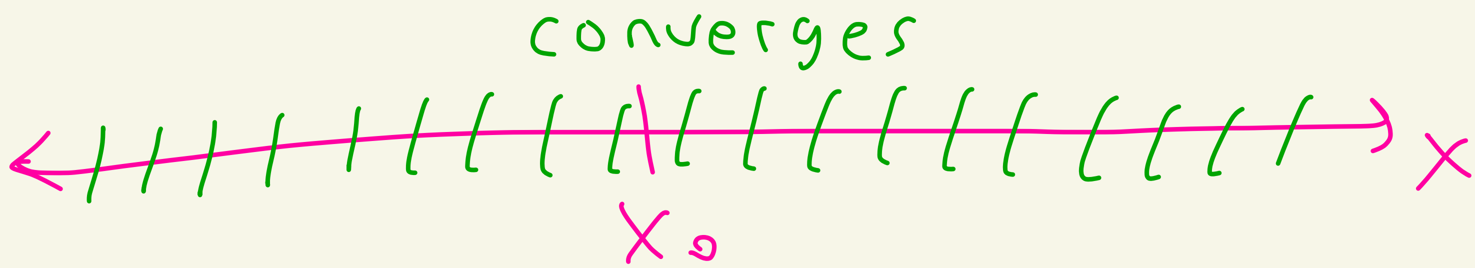


It will diverge when $x < x_0 - r$ or $x_0 + r < x$.

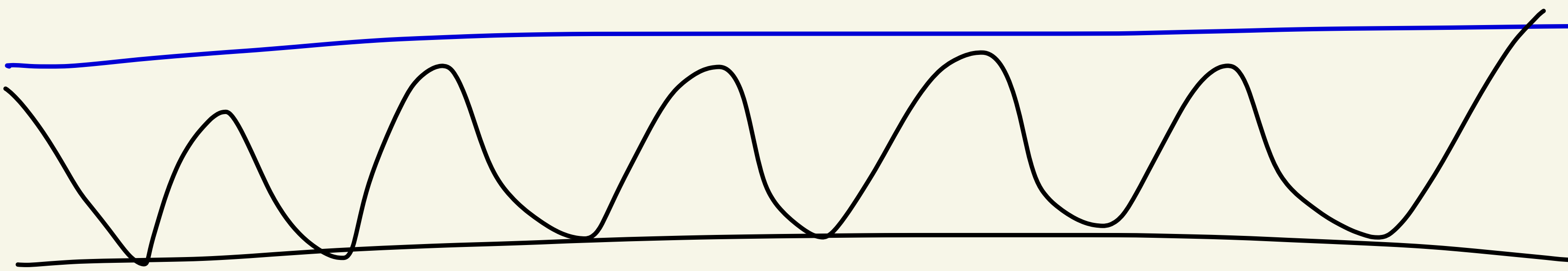
It may converge or diverge at the endpoints $x = x_0 - r$, $x = x_0 + r$.

r is called the radius of convergence

③ the series converges for all x.



Here the radius of convergence is $r = \infty$



Ex: From Calc II:

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

These are both centered at $x_0 = 0$ and have radius of convergence $r = \infty$

For example:

$$\sin(2) = 2 - \frac{1}{3!}(2)^3 + \frac{1}{5!}(2)^5 - \frac{1}{7!}(2)^7 + \cdots$$

$$= 2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \cdots$$