Math 2150-01

10/22/25

Topic 10 – Reduction of order

Suppose you know one solution y_1 to the homogeneous ODE

$$\boxed{y'' + a_1(x)y' + a_0(x)y = 0} \quad (*)$$

on an interval I where $y_1(x) \neq 0$ for all x in I. Then one can find another solution using

$$\boxed{y_2 = y_1 \left(\int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx \right)}$$

Further, y_1 and y_2 will be linearly independent on I. Thus,

$$y_h = c_1 y_1 + c_2 y_2$$

will give all solutions to $(*)$ on I.

Ex: Consider

$$\boxed{(x^2+1)y'' - 2xy' + 2y = 0 \quad (**)}$$

on $I = (0, \infty)$

Let $y_1 = x$. $\leftarrow$ $\boxed{\text{Note } y_1 \neq 0 \text{ on } I}$

Note that y_1 solves $(**)$ because plugging it into $(**)$ gives

$$(x^2+1)\underbrace{(0)}_{y_1''=0} - 2x\underbrace{(1)}_{y_1'=1} + 2\underbrace{(x)}_{y_1=x}$$

$$= 0 - 2x + 2x$$

$$= 0$$

Let's find y_2 using our formula.

First we divide (**) by (x^2+1) to put a 1 in front of y''.

We get:

$$y'' - \frac{2x}{(x^2+1)} y' + \frac{2}{(x^2+1)} y = 0$$

We get

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\, dx$$

$$= x \int \frac{e^{-\int \left(-\frac{2x}{x^2+1}\right) dx}}{x^2}\, dx$$

$$= x\left[\int \frac{e^{\int \frac{2x}{x^2+1}dx}}{x^2}\,dx\right]$$

$$\int \frac{2x}{x^2+1}\,dx = \int \frac{1}{u}\,du = \ln|u|$$

$$u = x^2+1$$
$$du = 2x\,dx$$

$$= \ln|x^2+1| = \ln(x^2+1)$$

$$x^2+1>0$$

$$= x\left[\int \frac{e^{\ln(x^2+1)}}{x^2}\,dx\right]$$

$$= x\left[\int \frac{x^2+1}{x^2}\,dx\right]$$

$$\boxed{e^{\ln(z)} = z}$$

$$= x\left[\int \left(\frac{x^2}{x^2} + \frac{1}{x^2}\right) dx\right]$$

$$= x\int \left(1 + x^{-2}\right) dx$$

$$= x\left[x + \frac{x^{-1}}{-1}\right]$$

$$= x\left[x - \frac{1}{x}\right]$$

$$= x^2 - 1$$

Summary: $y_1 = x$, $y_2 = x^2 - 1$

are linearly independent solutions to

$$(x^2+1)y'' - 2xy' + 2y = 0$$

on $I = (0, \infty)$

The general solution is

$$y_h = c_1 x + c_2 (x^2 - 1)$$

Where c_1, c_2 are constants.

Ex: Given that $y_1 = x^4$ is a solution to

$$x^2 y'' - 7xy' + 16y = 0$$

on $I = (0, \infty)$, find the general solution.

Divide by x^2 to get

$$y'' - \frac{7x}{x^2} y' + \frac{16}{x^2} y = 0$$

which is

$$y'' \underbrace{- \frac{7}{x}}_{a_1 = -\frac{7}{x}} y' + \frac{16}{x^2} y = 0$$

We get

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx$$

$$= x^4 \int \frac{e^{-\int -\frac{7}{x}\,dx}}{(x^4)^2}\,dx$$

$$= x^4 \int \frac{e^{7\int \frac{1}{x}\,dx}}{x^8}\,dx$$

$$= x^4 \int \frac{e^{7\ln|x|}}{x^8}\,dx$$

$$= x^4 \int \frac{e^{7\ln(x)}}{x^8}\,dx$$

$I = (0,\infty)$

$x > 0$

$|x| = x$

$$= x^4 \int \frac{e^{\ln(x^7)}}{x^8}\,dx$$

$c\ln(z) = \ln(z^c)$

$e^{\ln(z)} = z$

$$= x^4 \int \frac{x^7}{x^8}\,dx$$

$$= x^4 \int \frac{1}{x}\, dx$$

$$= x^4 \ln|x|$$

$$= x^4 \ln(x)$$

$I = (0, \infty)$

$x > 0$

$|x| = x$

Thus, the general solution is

$$y_h = c_1 y_1 + c_2 y_2$$

$$= c_1 x^4 + c_2 x^4 \ln(x)$$

Topic 11 – Review of power series

<u>Def:</u> An <u>infinite sum/series</u> is a sum of the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + a_4 + \cdots$$

[The sum doesn't have to start at $n=0$]

What does the above sum mean?

We define partial sums:

$$S_N = a_0 + a_1 + a_2 + \cdots + a_N$$

For example,

$$s_0 = a_0$$
$$s_1 = a_0 + a_1$$
$$s_2 = a_0 + a_1 + a_2$$
$$s_3 = a_0 + a_1 + a_2 + a_3$$
$$\vdots \qquad \vdots$$

Define

$$\sum_{n=0}^{\infty} a_n = \lim_{N \to \infty} s_N$$
$$= \lim_{N \to \infty} (a_0 + a_1 + a_2 + \dots + a_N)$$

if the limit exists.

If $\lim_{N \to \infty} s_N = L$, then we write $\sum_{n=0}^{\infty} a_n = L$

and say $\sum_{n=0}^{\infty} a_n$ converges.

If the limit doesn't exist then we say $\sum_{n=0}^{\infty} a_n$ diverges.

<u>Ex:</u> Consider

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

Let's make a table of partial sums.

N	$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^N}$
0	1

1	$1+\frac{1}{2}=1.5$
2	$1+\frac{1}{2}+\frac{1}{2^2}=1.75$
3	$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}=1.875$
4	$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}=1.9375$
5	1.96875
$\vdots$	
50	$1.\underbrace{999\ldots99}_{15\ 9\text{'s}}11182\ldots$
$\vdots$	$\vdots$
100	$1.\underbrace{999\ldots99}_{30\ 9\text{'s}}11139\ldots$

In Calc II you show

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2$$

Geometric Sum

If $-1 < r < 1$, then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$