Math 2150-01
10/20/25

Ex: Solve

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Step 1: Solve $y'' - 4y' + 4y = 0$

The characteristic poly. is

$$r^2 - 4r + 4 = 0$$

$$(r-2)(r-2) = 0$$

$$r = 2, 2$$

So,

$$y_h = c_1 \underbrace{\boxed{e^{2x}}}_{y_1} + c_2 \underbrace{\boxed{xe^{2x}}}_{y_2}$$

Step 2: Find y_p for

$$y'' - 4y' + 4y = \boxed{(x+1)e^{2x}}$$

$b(x)$

Use: $y_p = v_1 y_1 + v_2 y_2$

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)}\,dx, \quad v_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)}\,dx$$

Let's calculate $W(y_1, y_2)$ first.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$

$$= (e^{2x})(e^{2x} + 2xe^{2x}) - (2e^{2x})(xe^{2x})$$

$$= e^{4x} + 2xe^{4x} - 2xe^{4x}$$

$$= e^{4x}$$

We have

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)}\, dx$$

$$= \int \frac{-(xe^{2x})(x+1)e^{2x}}{e^{4x}}\, dx$$

$$= \int \frac{-x(x+1)e^{2x}e^{2x}}{e^{4x}}\, dx$$

$$= \int \frac{(-x^2 - x)\cancel{e^{4x}}}{\cancel{e^{4x}}}\,dx$$

$$= \int (-x^2 - x)\,dx$$

$$= \boxed{-\frac{1}{3}x^3 - \frac{1}{2}x^2}$$

v_1

And

$$v_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)}\,dx$$

$$= \int \frac{\cancel{e^{2x}}(x+1)\cancel{e^{2x}}}{\cancel{e^{4x}}}\,dx$$

$e^{2x}e^{2x} = e^{4x}$

$$= \int (x+1)\,dx = \boxed{\frac{1}{2}x^2 + x} \leftarrow \boxed{v_2}$$

Thus,

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x}$$

Step 3: The general solution to

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

is

$$y = y_h + y_p$$

$$= c_1 e^{2x} + c_2 x e^{2x} + \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x}$$

Ex: Solve $y'' + y = \tan(x)$

Step 1: Solve $y'' + y = 0$

The characteristic poly is

$$r^2 + 1 = 0$$

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{\pm\sqrt{-4}}{2} = \frac{\pm\sqrt{4}\sqrt{-1}}{2}$$

$$= \frac{\pm 2i}{2} = \pm i$$

$$= \underbrace{0 \pm 1 \cdot i}_{\alpha \pm \beta i}$$

$\alpha = 0, \beta = 1$

So,

$$y_h = c_1 e^{0x} \cos(1 \cdot x) + c_2 e^{0x} \sin(1 \cdot x)$$

$$\underbrace{}_{c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)}$$

$$= c_1 \underbrace{\boxed{\cos(x)}}_{y_1} + c_2 \underbrace{\boxed{\sin(x)}}_{y_2}$$

Step 2: Find y_p

for $y'' + y = \underbrace{\boxed{\tan(x)}}_{b(x)}$

First the Wronskian!

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

$$= (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

$$= \cos^2(x) + \sin^2(x)$$

$$= 1$$

Formula time.

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)} dx$$

$$= \int \frac{-\sin(x) \tan(x)}{1} dx$$

$$= -\int \sin(x) \cdot \frac{\sin(x)}{\cos(x)}\,dx$$

$$= -\int \frac{\sin^2(x)}{\cos(x)}\,dx$$

$\sin^2(x) + \cos^2(x) = 1$

$\sin^2(x) = 1 - \cos^2(x)$

$$= -\int \frac{1-\cos^2(x)}{\cos(x)}\,dx$$

$$= -\int \frac{1}{\cos(x)}\,dx + \int \frac{\cos^2(x)}{\cos(x)}\,dx$$

$$= -\int \sec(x)\,dx + \int \cos(x)\,dx$$

V_1

$$= \boxed{-\ln\left|\sec(x) + \tan(x)\right| + \sin(x)}$$

And

$$v_2 = \int \frac{y_1\, b(x)}{W(y_1, y_2)}\, dx$$

$$= \int \frac{\cos(x)\tan(x)}{1}\, dx$$

$$= \int \cos(x) \cdot \frac{\sin(x)}{\cos(x)}\, dx$$

$$= \int \sin(x)\, dx$$

$$= \boxed{-\cos(x)} \leftarrow \boxed{v_2}$$

So,

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \left[-\ln\left|\sec(x) + \tan(x)\right| + \sin(x)\right]\cos(x)$$

$$+(-\cos(x))\sin(x)$$

$$= \boxed{-\cos(x) \cdot \ln|\sec(x) + \tan(x)|}$$

Step 3: The general solution to $y'' + y = \tan(x)$ is

$$y = y_h + y_p$$

$$= c_1\cos(x) + c_2\sin(x) - \cos(x)\ln|\sec(x) + \tan(x)|$$
