Math 2150-01
10/15/25

Topic 9 - Variation of parameters

This section gives another way to find y_p.
It will work in situations where topic 8 doesn't work such as

$$y'' + y = \tan(x)$$

Also topic 8 only applies to equations with constant coefficients.
Topic 9 will work for any second order linear equation such as

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

Let's derive the formula for topic 9

Suppose you already have the general solution to the linear homogeneous ODE

$$\boxed{y'' + a_1(x)\, y' + a_0(x)\, y = 0} \quad (*)$$

and it is

$$y_h = c_1 y_1 + c_2 y_2$$

where y_1 and y_2 are linearly independent solutions to $(*)$.

From this we can find a particular solution y_p to

$$\boxed{y'' + a_1(x)y' + a_0(x)y = b(x)} \quad (**)$$

To do this, set

$$y_p = v_1 y_1 + v_2 y_2$$

(y_1, y_2 are from above)

where v_1, v_2 are unknown functions to be determined. We will plug y_p into $(**)$ and find v_1, v_2 to make y_p solve $(**)$.

We need y_p, y_p', and y_p''.

We have

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_p' = (v_1' y_1 + v_1 y_1') + (v_2' y_2 + v_2 y_2')$$

$$= (v_1 y_1' + v_2 y_2') + \underbrace{(v_1' y_1 + v_2' y_2)}_{\text{(assume this is 0)}}$$

We will assume

$$\boxed{v_1' y_1 + v_2' y_2 = 0} \leftarrow \boxed{\text{Condition 1}}$$

Then

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_p' = v_1 y_1' + v_2 y_2'$$

$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

Plug these into

$$y'' + a_1(x) y' + a_0(x) y = b(x) \leftarrow \boxed{(**)}$$

to get

$$\left(v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''\right) \longleftarrow y_p''$$

$$+ a_1(x)\left[v_1 y_1' + v_2 y_2'\right] \longleftarrow + a_1(x) y_p'$$

$$+ a_0(x)\left[v_1 y_1 + v_2 y_2\right] \longleftarrow + a_0(x) y_p$$

$$= b(x) \longleftarrow = b(x)$$

This becomes

$$v_1\left(\underbrace{y_1'' + a_1(x) y_1' + a_0(x) y_1}_{0}\right)$$

$$+ v_2\left(\underbrace{y_2'' + a_1(x) y_2' + a_0(x) y_2}_{0}\right)$$

$$+ \left(v_1' y_1' + v_2' y_2'\right) = b(x)$$

[The above are 0 because y_1, y_2 solve the homogeneous eqn (*)]

We are left with

$$\boxed{v_1' y_1' + v_2' y_2' = b(x)}$$ ← condition 2

Summary so far:

In order for $y_p = v_1 y_1 + v_2 y_2$ to solve (**) we need

$$\boxed{\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \quad (1) \\ v_1' y_1' + v_2' y_2' &= b(x) \quad (2) \end{aligned}}$$

In (1), (2) the unknowns are v_1' and v_2'.

To solve for v_2' we can calculate $y_1' * (1) - y_1 * (2)$ to get:

$$(y_1' \cancel{v_1' y_1} + y_1' v_2' y_2) - (y_1 \cancel{v_1' y_1'} + y_1 v_2' y_2') = y_1' \cdot 0 - y_1 \cdot b(x)$$

We get

$$y_1' v_2' y_2 - y_1 v_2' y_2' = -y_1 b(x)$$

Thus,

$$v_2' (y_1' y_2 - y_1 y_2') = -y_1 b(x)$$

So,

$$v_2' = \frac{-y_1 b(x)}{y_1' y_2 - y_1 y_2'}$$

So,

$$v_2' = \frac{-y_1 b(x)}{-(y_1 y_2' - y_1' y_2)}$$

Thus,

$$v_2' = \frac{y_1 b(x)}{y_1 y_2' - y_1' y_2}$$

$$\boxed{\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= y_1 y_2' - y_1' y_2 \end{aligned}}$$

So,

$$v_2' = \frac{y_1 b(x)}{W(y_1, y_2)}$$

Hence

$$v_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)} \, dx$$

Similarly you can solve for v_1 by calculating $y_2' * ① - y_2 * ②$.

You will get

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)} \, dx$$

Summary:

Suppose that y_1 and y_2 are two linearly independent solutions to

$$\boxed{y'' + a_1(x)y' + a_0(x)y = 0}$$

Then a particular solution to

$$\boxed{y'' + a_1(x)y' + a_0(x)y = b(x)}$$

is given by

$$y_p = v_1 y_1 + v_2 y_2$$

where

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)}\,dx, \quad v_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)}\,dx$$