Math 2150-01
10/1/25

Topic 8 continued...

Ex: Solve
$$y'' - y' + y = 2\sin(3x)$$

Step 1: We did this on Monday.

The general solution to
$$y'' - y' + y = 0$$
homogeneous equation

is
$$y_h = c_1 e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

Step 2: We want a particular

solution y_p to

$$y'' - y' + y = \boxed{2\sin(3x)}$$

$h(x)$

From our table we guess

$$y_p = A\cos(3x) + B\sin(3x)$$

Let's plug it in. We have:

$$y_p' = -3A\sin(3x) + 3B\cos(3x)$$

$$y_p'' = -9A\cos(3x) - 9B\sin(3x)$$

Now plug them into

to get:

$$\left.\begin{aligned}&(-9A\cos(3x) - 9B\sin(3x))\\ -&(-3A\sin(3x) + 3B\cos(3x))\\ +&(A\cos(3x) + B\sin(3x))\end{aligned}\right]\begin{aligned}&y_p''\\ -&y_p'\\ +&y_p\end{aligned}$$

$= 2\sin(3x)$

Thus,

$$\underbrace{[-8A-3B]}_{0}\cos(3x)+\underbrace{[3A-8B]}_{2}\sin(3x)$$

$$= 2\sin(3x)$$

We need

$$\boxed{\begin{aligned} -8A-3B&=0 \quad ① \\ 3A-8B&=2 \quad ② \end{aligned}}$$

① gives $A = -\frac{3}{8}B$.

Plug into ② to get: $3\overbrace{\left(-\frac{3}{8}B\right)}^{A}-8B=2$

Then, $-\frac{73}{8}B=2$

So, $B = 2\left(-\frac{8}{73}\right) = \boxed{\frac{-16}{73}}$

Then, $A = -\frac{3}{8}B = \left(-\frac{3}{8}\right)\left(\frac{-16}{73}\right) = \boxed{\frac{6}{73}}$

Hence,

$$\boxed{y_p = \frac{6}{73}\cos(3x) - \frac{16}{73}\sin(3x)}$$

Step 3: The general solution to

$$y'' - y' + y = 2\sin(3x)$$

is

$$\boxed{\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right) \\ &\quad + \frac{6}{73}\cos(3x) - \frac{16}{73}\sin(3x) \end{aligned}}$$

Ex: Solve $y'' + 3y = xe^{3x}$

Step 1: Solve the homogeneous equation $y'' + 3y = 0$.

We want the roots to

$$r^2 + 3 = 0$$

The roots are

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{\pm\sqrt{-12}}{2} = \pm \frac{\sqrt{12}\sqrt{-1}}{2}$$

$\boxed{i = \sqrt{-1}}$

$$= \pm \frac{2\sqrt{3}}{2} i$$

$$= \pm \sqrt{3}\, i$$

$$= \underbrace{0 \pm \sqrt{3}\, i}_{\alpha \pm \beta i}$$

$$\alpha = 0, \ \beta = \sqrt{3}$$

Thus,

$$y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

$$= c_1 e^{0x} \cos(\sqrt{3}x) + c_2 e^{0x} \sin(\sqrt{3}x)$$

$$= c_1 \cos(\sqrt{3}\, x) + c_2 \sin(\sqrt{3}\, x)$$

$$\boxed{e^{0x} = e^0 = 1}$$

<u>Step 2:</u> Find a particular solution y_p to $y'' + 3y = xe^{3x}$

We guess

$$y_p = (Ax+B)e^{3x} = \boxed{Axe^{3x} + Be^{3x}}$$

Let's find the derivatives.

$$y_p' = Ae^{3x} + Ax(3e^{3x}) + 3Be^{3x}$$

$$= Ae^{3x} + 3Axe^{3x} + 3Be^{3x}$$

And

$$y_p'' = 3Ae^{3x} + (3Ae^{3x} + 3Ax(3e^{3x}))$$
$$+ 9Be^{3x}$$

$$= 6Ae^{3x} + 9Axe^{3x} + 9Be^{3x}$$

Plug this into $y'' + 3y = xe^{3x}$ to get:

$$\underbrace{\left(6Ae^{3x} + 9Axe^{3x} + 9Be^{3x}\right)}_{y_p''}$$

$$+ 3\underbrace{\left(Be^{3x} + Axe^{3x}\right)}_{y_p} = xe^{3x}$$

Combine like terms:

$$\underbrace{(6A+12B)}_{0}e^{3x} + \underbrace{(12A)}_{1}xe^{3x} = xe^{3x}$$

Need

$$\boxed{\begin{aligned} 6A+12B &= 0 \quad (1) \\ 12A &= 1 \quad (2) \end{aligned}}$$

(2) gives $\boxed{A = \frac{1}{12}}$

Plug into (1) to get $6\overbrace{\left(\frac{1}{12}\right)}^{A} + 12B = 0$

We get $\frac{1}{2} + 12B = 0$.

So, $B = -\frac{1}{2} \cdot \frac{1}{12} = \boxed{-\frac{1}{24}}$

Thus,

$$\boxed{y_p = (Ax+B)e^{3x} = \left(\frac{x}{12} - \frac{1}{24}\right)e^{3x}}$$

Step 3: The general solution to $y'' + 3y = xe^{3x}$ is

$$\boxed{\begin{aligned} y &= y_h + y_p \\ &= c_1 \cos(\sqrt{3}\,x) + c_2 \sin(\sqrt{3}\,x) \\ &\quad + \left(\frac{x}{12} - \frac{1}{24}\right)e^{3x} \end{aligned}}$$