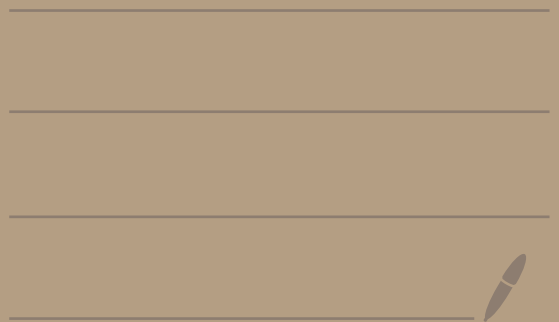


Math 2150-01

10/1/25



Topic 8 continued...

Ex: Solve

$$y'' - y' + y = 2 \sin(3x)$$

Step 1: We did this on Monday.

The general solution to

$$y'' - y' + y = 0$$

← homogeneous equation

is

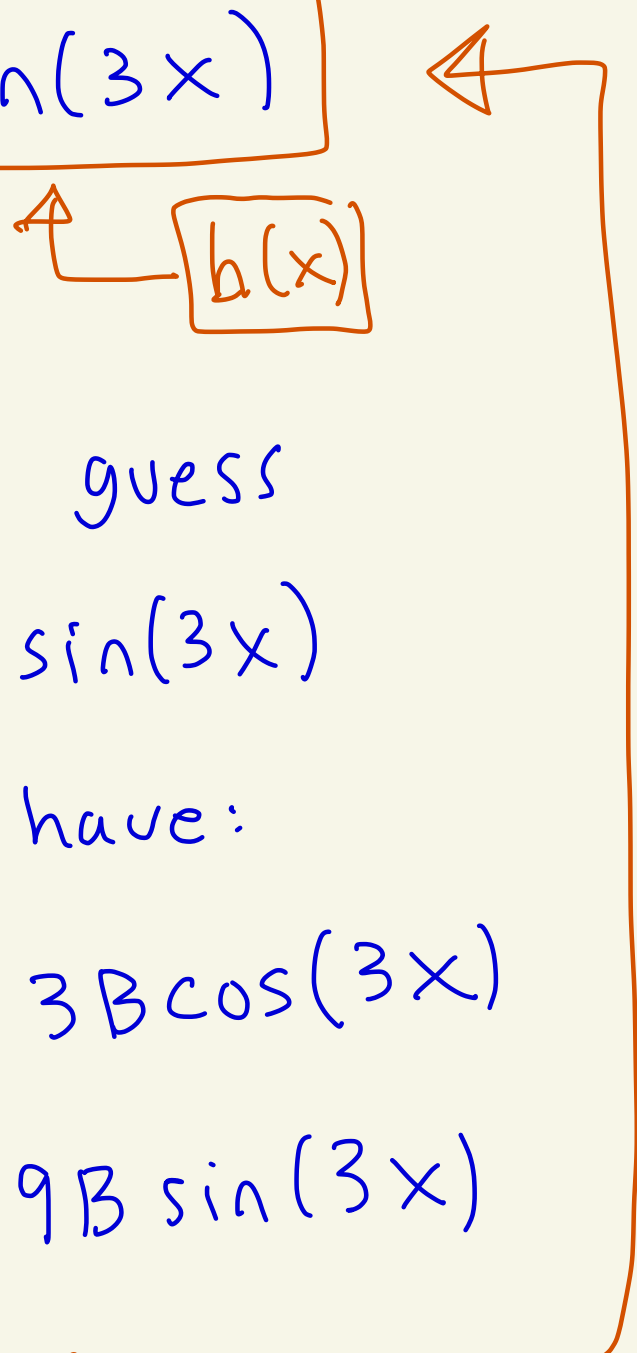
$$y_h = c_1 e^{x/2} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{x/2} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

Step 2: We want a particular

solution y_p to

$$y'' - y' + y = \boxed{2 \sin(3x)}$$

$\uparrow$ $\boxed{b(x)}$



From our table we guess

$$y_p = A \cos(3x) + B \sin(3x)$$

Let's plug it in. We have:

$$y_p' = -3A \sin(3x) + 3B \cos(3x)$$

$$y_p'' = -9A \cos(3x) - 9B \sin(3x)$$

Now plug them into
to get:

$$\left[\begin{array}{l} (-9A \cos(3x) - 9B \sin(3x)) \\ -(-3A \sin(3x) + 3B \cos(3x)) \\ + (A \cos(3x) + B \sin(3x)) \end{array} \right] \begin{array}{l} y_p'' \\ - y_p' \\ + y_p \end{array}$$

$$= 2 \sin(3x)$$

Thus,

$$\underbrace{[-8A - 3B]}_0 \cos(3x) + \underbrace{[3A - 8B]}_2 \sin(3x) = 2 \sin(3x)$$

We need

$$\begin{array}{l} -8A - 3B = 0 \quad (1) \\ 3A - 8B = 2 \quad (2) \end{array}$$

① gives $A = -\frac{3}{8}B$.

Plug into ② to get: $3 \overbrace{\left(-\frac{3}{8}B\right)}^A - 8B = 2$

Then, $-\frac{73}{8}B = 2$

$$\text{So, } B = 2\left(-\frac{8}{73}\right) = \boxed{-\frac{16}{73}}$$

$$\text{Then, } A = -\frac{3}{8}B = \left(-\frac{3}{8}\right)\left(-\frac{16}{73}\right) = \boxed{\frac{6}{73}}$$

Hence,

$$y_p = \frac{6}{73} \cos(3x) - \frac{16}{73} \sin(3x)$$

Step 3: The general solution to

$$y'' - y' + y = 2 \sin(3x)$$

is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{x/2} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{x/2} \sin\left(\frac{\sqrt{3}}{2}x\right) \\ &\quad + \frac{6}{73} \cos(3x) - \frac{16}{73} \sin(3x) \end{aligned}$$

Ex: Solve $y'' + 3y = x e^{3x}$

Step 1: Solve the homogeneous equation $y'' + 3y = 0$.

We want the roots to

$$r^2 + 3 = 0$$

The roots are

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{\pm \sqrt{-12}}{2} = \pm \frac{\sqrt{12} \sqrt{-1}}{2}$$

$\boxed{\bar{\lambda} = \sqrt{-1}} \rightarrow$

$$= \pm \frac{2\sqrt{3}}{2} i$$

$$= \pm \sqrt{3} i$$

$$= \underbrace{0 \pm \sqrt{3} i}_{\alpha \pm \beta i}$$

$$\alpha = 0, \beta = \sqrt{3}$$

Thus,

$$y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

$$= c_1 e^{0x} \cos(\sqrt{3}x) + c_2 e^{0x} \sin(\sqrt{3}x)$$

$$= c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)$$

↑

$$e^{0x} = e^0 = 1$$

Step 2: Find a particular
solution y_p to $y'' + 3y = x e^{3x}$

We guess

$$y_p = (Ax + B)e^{3x} = \boxed{Ax e^{3x} + B e^{3x}}$$

Let's find the derivatives.

$$\begin{aligned} y_p' &= A e^{3x} + Ax(3e^{3x}) + 3B e^{3x} \\ &= A e^{3x} + 3Ax e^{3x} + 3B e^{3x} \end{aligned}$$

And

$$\begin{aligned} y_p'' &= 3A e^{3x} + (3A e^{3x} + 3Ax(3e^{3x})) \\ &\quad + 9B e^{3x} \end{aligned}$$

$$= 6A e^{3x} + 9Ax e^{3x} + 9B e^{3x}$$

Plug this into $y'' + 3y = x e^{3x}$
to get:

$$\underbrace{(6Ae^{3x} + 9Axe^{3x} + 9Be^{3x})}_{y_p''}$$

$$+ 3 \underbrace{(Be^{3x} + Axe^{3x})}_{y_p} = xe^{3x}$$

Combine like terms:

$$\underbrace{(6A + 12B)}_0 e^{3x} + \underbrace{(12A)}_1 xe^{3x} = xe^{3x}$$

Need

$$\boxed{\begin{array}{l} 6A + 12B = 0 \\ 12A = 1 \end{array}} \begin{array}{l} \textcircled{1} \\ \textcircled{2} \end{array}$$

$$\textcircled{2} \text{ gives } \boxed{A = \frac{1}{12}}$$

$$\text{Plug into } \textcircled{1} \text{ to get } 6 \overbrace{\left(\frac{1}{12}\right)}^A + 12B = 0$$

We get $\frac{1}{2} + 12B = 0$.

$$\text{So, } B = -\frac{1}{2} \cdot \frac{1}{12} = \boxed{-\frac{1}{24}}$$

Thus,

$$y_p = (Ax + B)e^{3x} = \left(\frac{x}{12} - \frac{1}{24}\right)e^{3x}$$

Step 3: The general solution
to $y'' + 3y = xe^{3x}$ is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x) \\ &\quad + \left(\frac{x}{12} - \frac{1}{24}\right)e^{3x} \end{aligned}$$
