① (a)

Let $y_1 = e^{2x}$ and $y_2 = e^{5x}$ and $I = (-\infty, \infty)$.

Then,

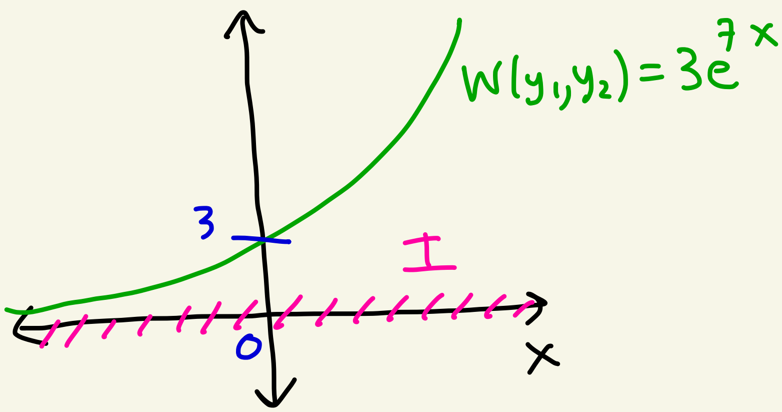
$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{5x} \\ 2e^{2x} & 5e^{5x} \end{vmatrix}$$

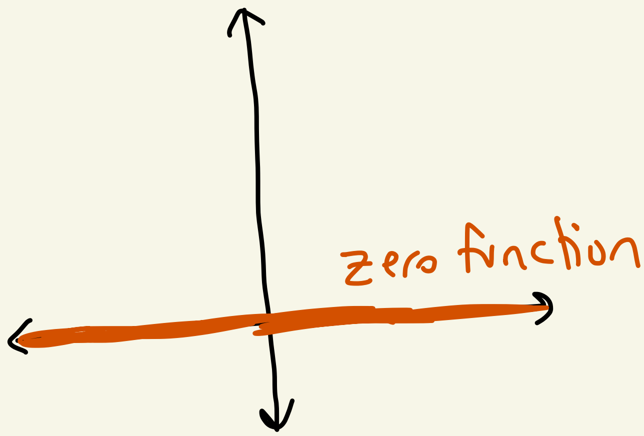
$$= (e^{2x})(5e^{5x}) - (e^{5x})(2e^{2x})$$

$$= 5e^{2x+5x} - 2e^{5x+2x}$$

$$= 3e^{7x}$$

We see that $W(y_1, y_2)$ is not the zero function on I, in particular $W(y_1, y_2)(0) = 3e^{7(0)} = 3e^{0} = 3 \neq 0$.





Thus, $y_1 = e^{2x}$ and $y_2 = e^{5x}$ are linearly independent on $I = (-\infty, \infty)$.

$\boxed{1 (b)}$

We have $y_1 = e^{2x}$, $y_1' = 2e^{2x}$, $y_1'' = 4e^{2x}$

Thus,

$$
\begin{aligned}
y_1'' - 7y_1' + 10y_1 &= 4e^{2x} - 7(2e^{2x}) + 10(e^{2x}) \\
&= 4e^{2x} - 14e^{2x} + 10e^{2x} \\
&= 0
\end{aligned}
$$

Thus, $y_1 = e^{2x}$ solves $y'' - 7y' + 10y = 0$.

We have $y_2 = e^{5x}$, $y_2' = 5e^{5x}$, $y_2'' = 25e^{5x}$.

Thus,

$$
\begin{aligned}
y_2'' - 7y_2' + 10y_2 &= 25e^{5x} - 7(5e^{5x}) + 10(e^{5x}) \\
&= 25e^{5x} - 35e^{5x} + 10e^{5x} \\
&= 0
\end{aligned}
$$

Thus, $y_1 = e^{5x}$ solves $y'' - 7y' + 10y = 0$.

$\boxed{①(c)}$ Since $y_1 = e^{2x}$ and $y_2 = e^{5x}$ are linearly independent solutions to $y'' - 7y' + 10y = 0$ on $I = (-\infty, \infty)$ we know that the general solution y_h to $y'' - 7y' + 10y = 0$ on I is

$$y_h = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{c_1 y_1 + c_2 y_2}$$

$\boxed{①(d)}$

We have $y_p = 6e^x$, $y_p' = 6e^x$, $y_p'' = 6e^x$

So,

$$\begin{aligned} y_p'' - 7y_p' + 10y_p &= 6e^x - 7(6e^x) + 10(6e^x) \\ &= 6e^x - 42e^x + 60e^x \\ &= 24e^x \end{aligned}$$

Thus, $y_p = 6e^x$ solves $y'' - 7y' + 10y = 24e^x$ on I

$\boxed{①(e)}$ The general solution to $y'' - 7y' + 10y = 24e^x$ on I is

$$y = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{y_h} + \underbrace{6e^x}_{y_p}$$

1(f)

We know that $y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$ is the general solution to $y'' - 7y' + 10y = 24e^{x}$ on I.
Let's find c_1, c_2 so our solution satisfies $y(0)=0$, $y'(0)=6$.

We have $y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$
$y' = 2c_1 e^{2x} + 5c_2 e^{5x} + 6e^{x}$

$$\boxed{\begin{aligned} y(0)&=0 \\ y'(0)&=6 \end{aligned}} \longrightarrow \boxed{\begin{aligned} c_1 e^{2(0)} + c_2 e^{5(0)} + 6e^{0} &= 0 \\ 2c_1 e^{2(0)} + 5c_2 e^{5(0)} + 6e^{0} &= 6 \end{aligned}}$$

$(e^{0}=1)$

$$\boxed{\begin{aligned} c_1 + c_2 + 6 &= 0 \\ 2c_1 + 5c_2 + 6 &= 6 \end{aligned}} \longrightarrow \boxed{\begin{aligned} c_1 + c_2 &= -6 \quad (1) \\ 2c_1 + 5c_2 &= 0 \quad (2) \end{aligned}}$$

(1) gives $c_1 = -6 - c_2$.
Plug this into (2) to get $2(-6-c_2) + 5c_2 = 0$.
So, $-12 - 2c_2 + 5c_2 = 0$.
So, $3c_2 = 12$.

Thus, $c_2 = 4$.

Plug back into ① to get $c_1 = -6 - 4 = -10$.

Plug back into

$$y = c_1 e^{2x} + c_2 e^{5x} + 6x$$

to get

$$y = -10e^{2x} + c_2 e^{5x} + 6x$$

solves

$$y'' - 7y' + 10y = 24e^{x}, \quad y'(0) = 6, \; y(0) = 0.$$

$\boxed{2(a)}$ Let $y_1 = x^2$ and $y_2 = x^4$

Then

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^4 \\ 2x & 4x^3 \end{vmatrix}$$

$$= (x^2)(4x^3) - (2x)(x^4)$$

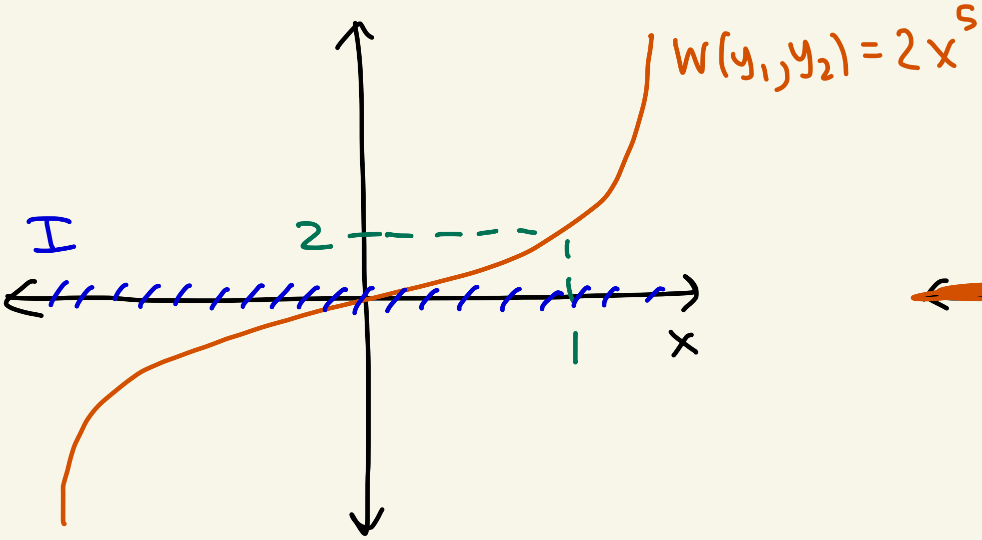
$$= 4x^{2+3} - 2x^{1+4}$$

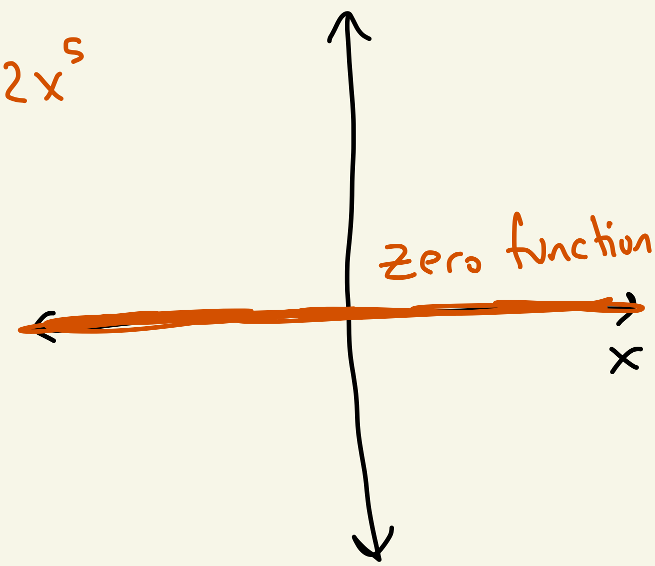
$$= 4x^5 - 2x^5$$

$$= 2x^5$$

This is not the zero function on $I = (-\infty, \infty)$.

For example, $W(y_1, y_2)(1) = 2(1)^5 = 2 \neq 0$





Thus, $y_1 = x^2$ and $y_2 = x^4$ are linearly independent on $I = (-\infty, \infty)$.

$\boxed{2(b)}$ We have

$$y_1 = x^2 \qquad y_2 = x^4$$
$$y_1' = 2x \qquad y_2' = 4x^3$$
$$y_1'' = 2 \qquad y_2'' = 12x^2$$

Thus,

$$\begin{aligned} x^2 y_1'' - 5x y_1' + 8y_1 &= x^2 \cdot (2) - 5x(2x) + 8x^2 \\ &= 2x^2 - 10x^2 + 8x^2 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} x^2 y_2'' - 5x y_2' + 8y_2 &= x^2 \cdot (12x^2) - 5x(4x^3) + 8x^4 \\ &= 12x^4 - 20x^4 + 8x^4 \\ &= 0 \end{aligned}$$

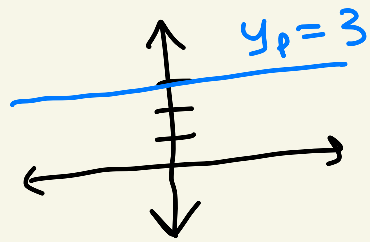
Thus, both $y_1 = x^2$ and $y_2 = x^4$ solve

$$x^2 y'' - 5xy' + 8y = 0$$

$\boxed{2(c)}$ Since $y_1 = x^2$, $y_2 = x^4$ are linearly independent solutions to the homogeneous linear equation $x^2 y'' - 5xy' + 8y = 0$ the general solution is

$$y_h = \underbrace{c_1 x^2 + c_2 x^4}_{c_1 y_1 + c_2 y_2}$$

where c_1, c_2 are constants.



$\boxed{2(d)}$ Let $y_p = 3$.

Then, $y_p' = 0$, $y_p'' = 0$.

Thus, plugging y_p into the left hand side of $x^2 y'' - 5xy' + 8y = 24$ gives

$$x^2 \cdot y_p'' - 5x y_p' + 8y_p = x^2(0) - 5x(0) + 8(3) = 24.$$

So, y_p is a particular solution to $x^2 y'' - 5xy' + 8y = 24$.

2 (e) The general solution to

$$x^2y''-5xy'+8y=24$$

on I is

$$y=\underbrace{c_1x^2+c_2x^4}_{y_h}+\underbrace{3}_{y_p}$$

---

2 (f) The general solution to

$$x^2y''-5xy'+8y=24$$

is given by

$$y=c_1x^2+c_2x^4+3$$

We want this solution to satisfy

$y'(1)=0$ and $y(1)=-1$

We have

$$y=c_1x^2+c_2x^4+3$$
$$y'=2c_1x+4c_2x^3$$

So, we must solve

$$\boxed{\begin{aligned} y(1)&=-1\\ y'(1)&=0\end{aligned}} \longleftrightarrow \boxed{\begin{aligned} c_1(1)^2+c_2(1)^4+3&=-1\\ 2c_1(1)+4c_2(1)^3&=0\end{aligned}} \longleftrightarrow \boxed{\begin{aligned} c_1+c_2&=-4 \quad (1)\\ 2c_1+4c_2&=0 \quad (2)\end{aligned}}$$

Solve for c_1 in ① to get $c_1 = -4 - c_2$.

Plug this into ② to get $2(-4-c_2)+4c_2=0$

This gives $-8-2c_2+4c_2=0$.

This gives $2c_2=8$.

So, $\boxed{c_2=4}$.

Thus, $\boxed{c_1 = -4 - c_2 = -4-4=-8}$

So the solution to

$$x^2y'' - 5xy' + 8y = 24, \quad y'(1)=0, \quad y(1)=-1$$

is given by

$$y = \underbrace{-8}_{c_1} \cdot x^2 + \underbrace{(-1)}_{c_2} \cdot x^4 + 3$$

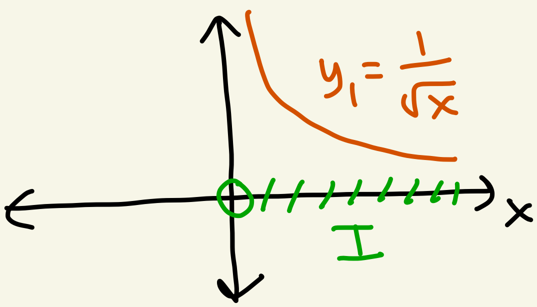
or

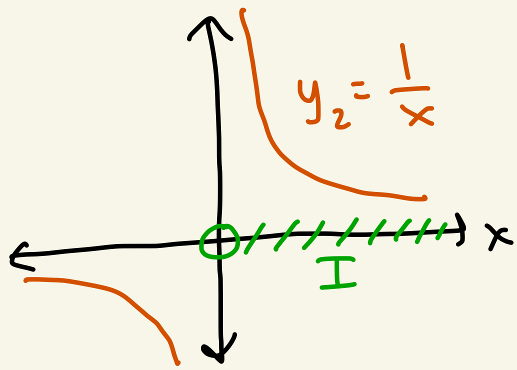
$$\boxed{y = -8x^2 - x^4 + 3}$$

Answer

③(a) Let $y_1 = x^{-1/2}$ and $y_2 = x^{-1}$

Note that $y_1 = \frac{1}{\sqrt{x}}$ and $y_2 = \frac{1}{x}$ are both defined on $I = (0, \infty)$ from their graphs:





We get that

$$W(f_1, f_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^{-1/2} & x^{-1} \\ -\frac{1}{2}x^{-3/2} & -x^{-2} \end{vmatrix}$$

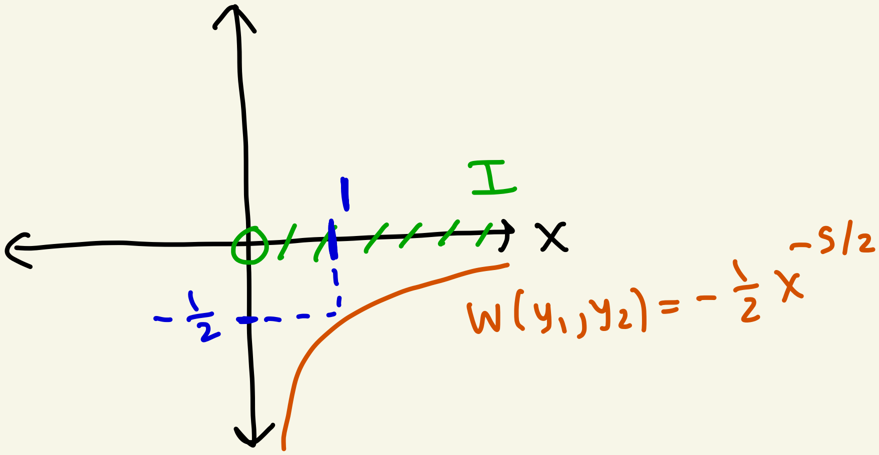
$$= (x^{-1/2})(-x^{-2}) - \left(-\frac{1}{2}x^{-3/2}\right)(x^{-1})$$

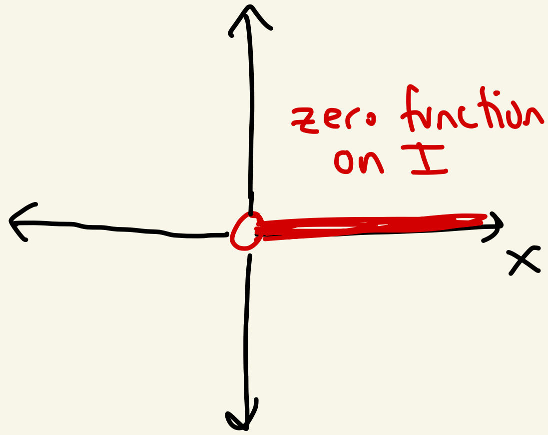
$$= -x^{-1/2-2} + \frac{1}{2}x^{-3/2-1}$$

$$= -x^{-5/2} + \frac{1}{2}x^{-5/2}$$

$$= -\frac{1}{2}x^{-5/2}$$

Note this isn't the zero function on $I = (0, \infty)$. In particular, $W(y_1, y_2)(1) = -\frac{1}{2}(1)^{-5/2} = -\frac{1}{2} \neq 0$.





Thus, $y_1 = x^{-1/2}$ and $y_2 = x^{-1}$ are linearly independent on $I = (0, \infty)$.

$\boxed{3(b)}$

We have that

$$\boxed{\begin{array}{ll} y_1 = x^{-1/2} & y_2 = x^{-1} \\ y_1' = -\frac{1}{2}x^{-3/2} & y_2' = -x^{-2} \\ y_1'' = \frac{3}{4}x^{-5/2} & y_2'' = 2x^{-3} \end{array}}$$

These are all defined when $x > 0$ that is on $I = (0, \infty)$.

Plugging y_1 and y_2 into the left side of

$$2x^2 y'' + 5xy' + y = 0$$

gives

$$2x^2 y_1'' + 5x y_1' + y_1 = 2x^2\left(\tfrac{3}{4}x^{-5/2}\right) + 5x\left(-\tfrac{1}{2}x^{-3/2}\right) + x^{-1/2}$$
$$= \tfrac{3}{2}x^{-1/2} - \tfrac{5}{2}x^{-1/2} + x^{-1/2}$$
$$= 0$$

and

$$2x^2 y_2'' + 5x y_2' + y_2 = 2x^2(2x^{-3}) + 5x(-x^{-2}) + x^{-1}$$
$$= 4x^{-1} - 5x^{-1} + x^{-1}$$
$$= 0$$

So, y_1 and y_2 both solve $2x^2y'' + 5xy' + y = 0$.

$\boxed{3(c)}$

The general solution to $2x^2y'' + 5xy' + y = 0$ is

$$y_h = \underbrace{c_1 x^{-1/2} + c_2 x^{-1}}_{c_1 y_1 + c_2 y_2}$$

where c_1, c_2 are constants.

3(d) Let $y_p = \frac{1}{15}x^2 - \frac{1}{6}x$.

Then, $y_p' = \frac{2}{15}x - \frac{1}{6}$

And, $y_p'' = \frac{2}{15}$.

Plugging y_p into $2x^2y'' + 5xy' + y = x^2 - x$ gives

$$2x^2y_p'' + 5xy_p' + y_p$$
$$= 2x^2\left(\frac{2}{15}\right) + 5x\left(\frac{2}{15}x - \frac{1}{6}\right) + \left(\frac{1}{15}x^2 - \frac{1}{6}x\right)$$
$$= \frac{4}{15}x^2 + \frac{10}{15}x^2 - \frac{5}{6}x + \frac{1}{15}x^2 - \frac{1}{6}x$$
$$= \frac{15}{15}x^2 - \frac{6}{6}x$$
$$= x^2 - x$$

So, y_p is a particular solution.

3(e) The general solution to

$$2x^2y'' + 5xy' + y = x^2 - x$$

is given by

$$y = \underbrace{c_1x^{-1/2} + c_2x^{-1}}_{y_h} \; + \; \underbrace{\frac{1}{15}x^2 - \frac{1}{6}x}_{y_p}$$

$\boxed{③(f)}$ We want

$$y = c_1 x^{-1/2} + c_2 x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x$$

where

$y'(1) = 0$ and $y(1) = 0$.

We have $y' = -\frac{1}{2}c_1 x^{-3/2} - c_2 x^{-2} + \frac{2}{15}x - \frac{1}{6}$

So we must solve

$$\boxed{\begin{matrix} y'(1)=0 \\ y(1)=0 \end{matrix}} \leftrightarrow \boxed{\begin{matrix} -\frac{1}{2}c_1(1)^{-3/2} - c_2(1)^{-2} + \frac{2}{15}(1) - \frac{1}{6} = 0 \\ c_1(1)^{-1/2} + c_2(1)^{-1} + \frac{1}{15}(1)^2 - \frac{1}{6}(1) = 0 \end{matrix}}$$

$$\updownarrow$$

$$\boxed{\begin{matrix} -\frac{1}{2}c_1 - c_2 = \frac{1}{30} & ① \\ c_1 + c_2 = \frac{1}{10} & ② \end{matrix}}$$

Solving ① for c_2 gives $c_2 = -\frac{1}{2}c_1 - \frac{1}{30}$.

Plug this into ② gives $c_1 + \left(-\frac{1}{2}c_1 - \frac{1}{30}\right) = \frac{1}{10}$.

So, $\frac{1}{2}c_1 = \frac{2}{15}$. Thus, $\boxed{c_1 = \frac{4}{15}}$.

c_2

And, $c_2 = -\frac{1}{2}c_1 - \frac{1}{30} = -\frac{1}{2}\left(\frac{4}{15}\right) - \frac{1}{30} = \frac{-5}{30} = \boxed{-\frac{1}{6}}$

So, the solution we are looking for is

$$\boxed{y = \frac{4}{15}x^{-1/2} - \frac{1}{6}x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x}$$