11/12
Tuesday

Test 2 Review

12.8

W2007
Quiz 3

② Find the local max/min's and saddle points of

$$f(x,y) = \frac{1}{3}x^3 - x + 3y^2$$

Step 1 Find the critical points.

$0 = f_x = x^2 - 1 \leftarrow \boxed{\begin{matrix} 0 = x^2 - 1 \\ x = \pm 1 \end{matrix}}$

$0 = f_y = 6y \leftarrow \boxed{\begin{matrix} 0 = 6y \\ 0 = y \end{matrix}}$

Critical points

$(x,y) = (1,0), (-1,0)$

Step 2

(a,b) is a critical point

$D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$

- $D(a,b) > 0,\ f_{xx}(a,b) > 0 \longrightarrow (a,b)$ local min
- $D(a,b) > 0,\ f_{xx}(a,b) < 0 \longrightarrow (a,b)$ local max
- $D(a,b) < 0 \longrightarrow (a,b)$ saddle point

$f_{xx} = 2x$

$f_{yy} = 6$

$f_{xy} = 0$

$D = (2x)(6) - 0^2$

$D = 12x$

$D(1,0) = 12(1) = 12 > 0$

$f_{xx}(1,0) = 2(1) = 2 > 0$

$(1,0)$ is a local min

$D(-1,0) = 12(-1) = -12 < 0$

$(-1,0)$ is a saddle point.

12.9

⑨ Find the max/min of

$f(x,y) = xy$

subject to $x^2+y^2-xy=9$

Use Lagrange's Method.

$f(x,y) = xy$

$g(x,y) = x^2+y^2-xy = 9$

$\nabla f = \langle y, x \rangle$

$\nabla g = \langle 2x-y, 2y-x \rangle$

$\boxed{\begin{array}{l} \nabla f = \lambda \nabla g \\ g = k \end{array}} \rightarrow \boxed{\begin{array}{l} \langle y, x \rangle = \lambda \langle 2x-y, 2y-x \rangle \\ x^2+y^2-xy=9 \end{array}}$

$\rightarrow \boxed{\begin{array}{ll} y = \lambda(2x-y) & (1) \\ x = \lambda(2y-x) & (2) \\ x^2+y^2-xy=9 & (3) \end{array}}$

① $y = \lambda(2x-y) \longrightarrow \lambda = \dfrac{y}{2x-y}$

② $x = \lambda(2y-x) \longrightarrow \lambda = \dfrac{x}{2y-x}$

Set ① & ② equal to each other.

$$\frac{y}{2x-y} = \lambda = \frac{x}{2y-x}$$

$$2y^2 - xy = 2x^2 - xy$$

$$y^2 = x^2 \leftarrow y^2 - x^2 = 0$$

$$(y-x)(y+x) = 0$$

$$\boxed{y = x \text{ or } y = -x}$$

If $2x-y=0$, then ① becomes $\boxed{y=0}$

So, $2x-y=0$ becomes $2x=0$ or $\boxed{x=0}$. This solves ① & ② but not ③ since ③ would be $0=9$ which isn't true.

So, $2x-y \neq 0$.

Same idea: If $2y-x=0$ then this forces $x=0$ & $y=0$ which doesn't solve ③

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(9

case 1: $y=x$

Plug $y=x$ into 3 to get

$$x^2+x^2-x^2=9$$
$$x^2=9$$
$$x=\pm 3$$

If $x=3$, $y=x=3$.

If $x=-3$, $y=x=-3$.

So we get

$$\boxed{(x,y)=(3,3),(-3,-3)}$$

case 2: $y=-x$

Plug $y=-x$ into 3 to get

$$x^2+(-x)^2-x(-x)=9$$
$$3x^2=9$$
$$x^2=3$$
$$x=\pm\sqrt{3}$$

If $x=\sqrt{3}$, then $y=-x=-\sqrt{3}$.

If $x=-\sqrt{3}$, then $y=-x=\sqrt{3}$.

So we get

$$\boxed{(x,y)=(\sqrt{3},-\sqrt{3}),(-\sqrt{3},\sqrt{3})}$$

Plug our points into $f(x,y)=xy$

$f(3,3)=(3)(3)=9$ max

$f(-3,-3)=(-3)(-3)=9$

$f(\sqrt{3},-\sqrt{3})=(\sqrt{3})(-\sqrt{3})=-3$ min

$f(-\sqrt{3},\sqrt{3})=(-\sqrt{3})(\sqrt{3})=-3$

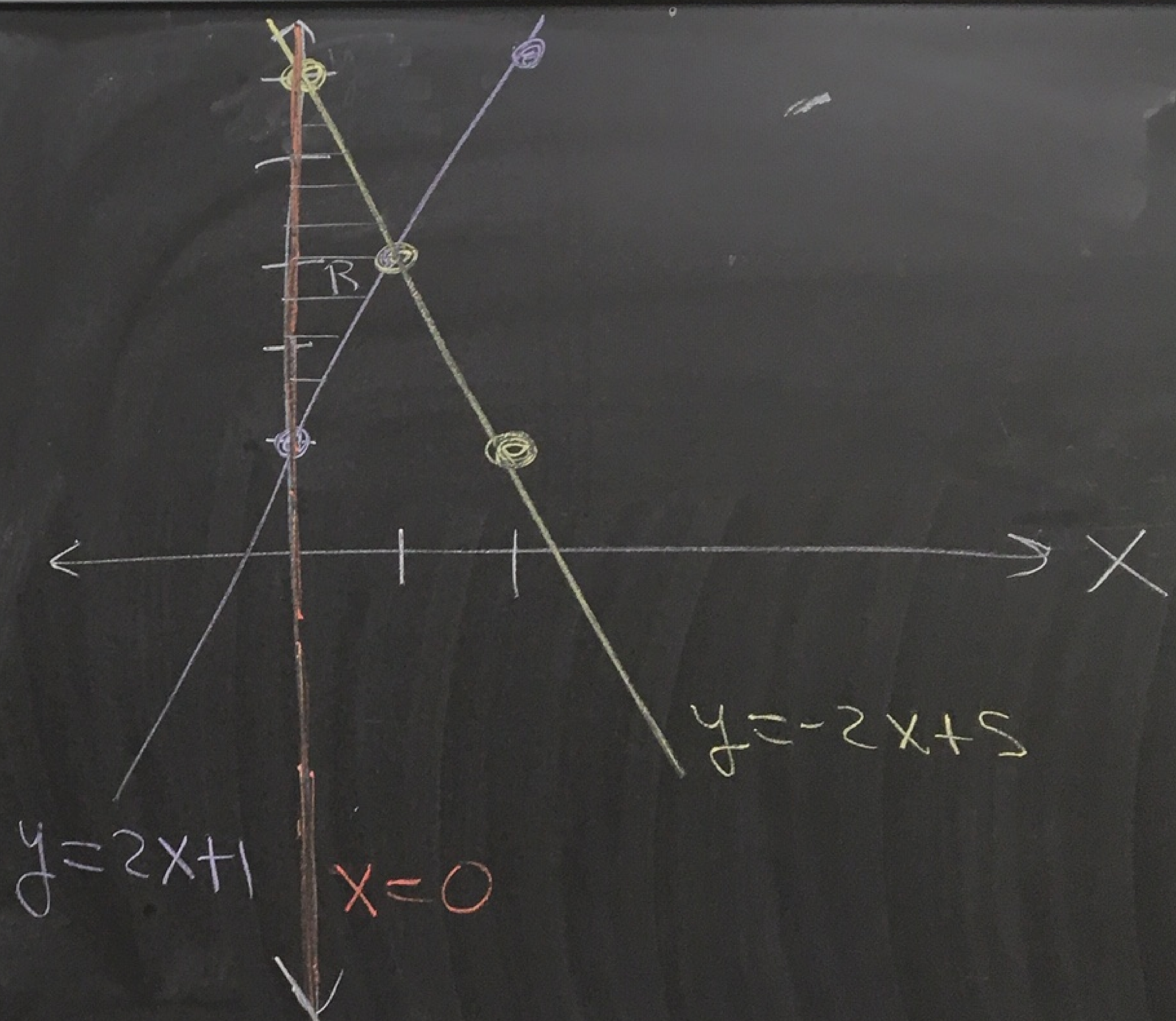
13.2

(27) $$\iint_R xy\,dA$$

R is bounded by

$x=0$, $y=2x+1$,

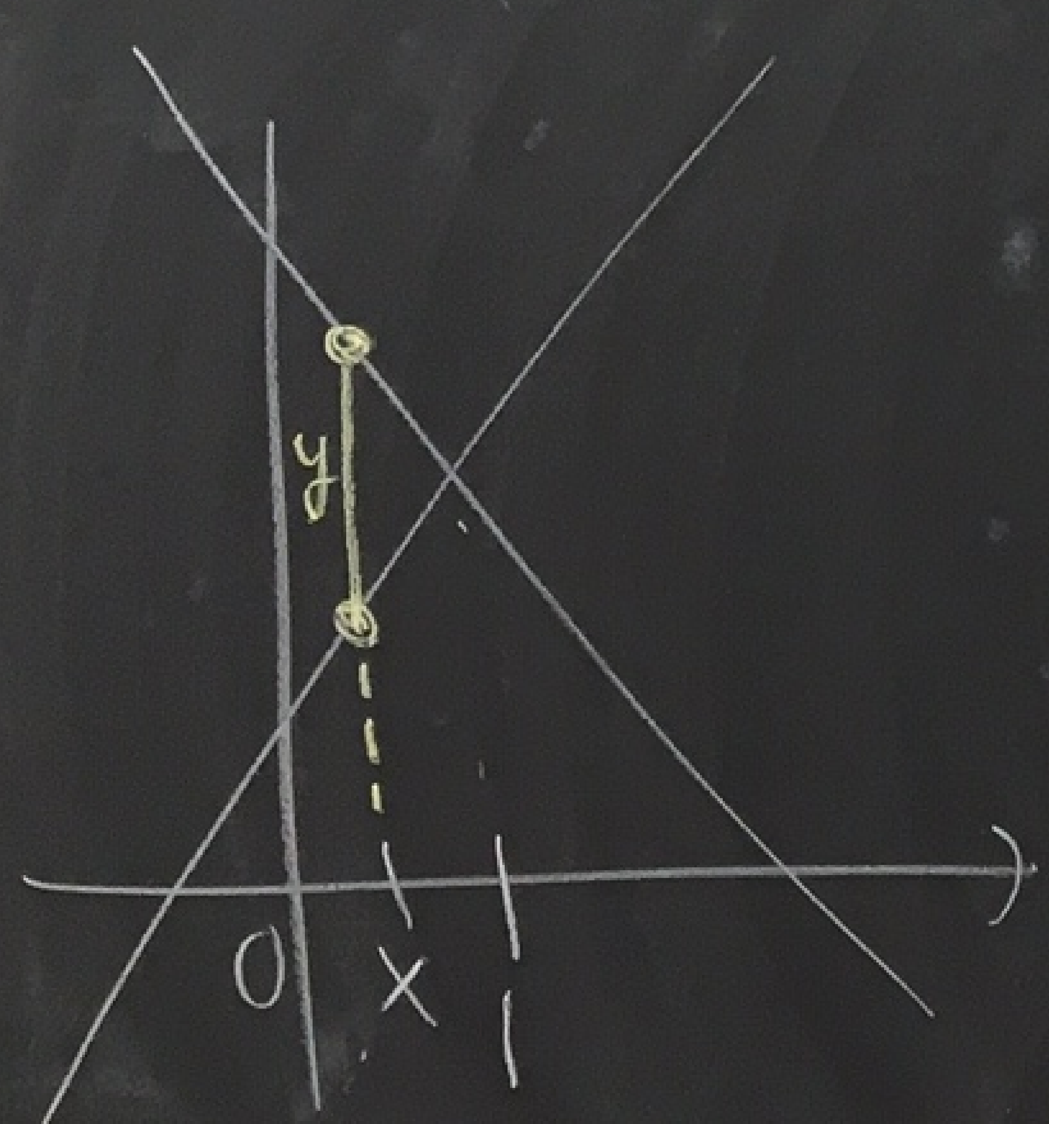
and $y=-2x+5$



① parametrize R

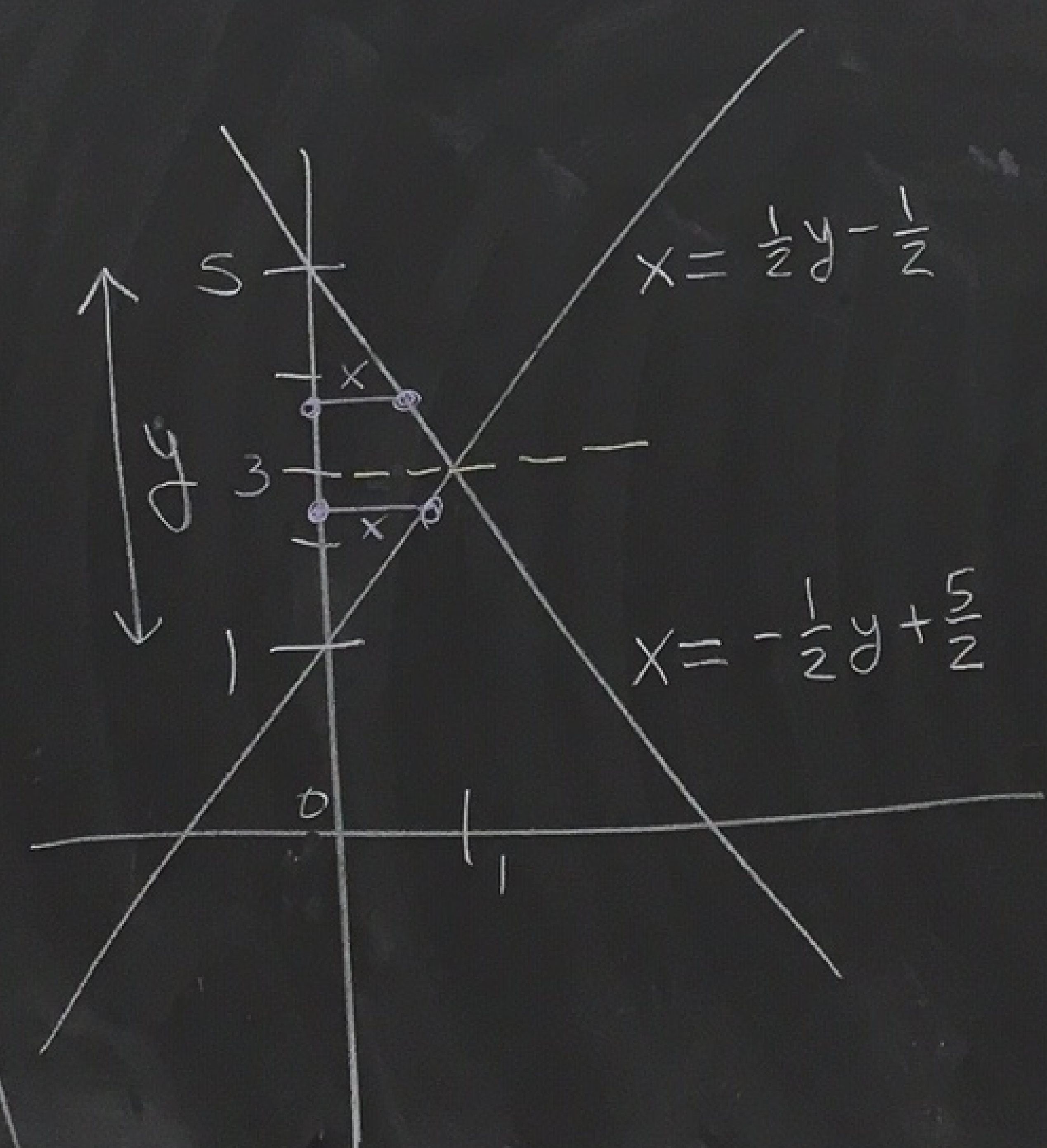
$$0 \le x \le 1$$

$$2x+1 \le y \le -2x+5$$



② parametrize R

$1 \le y \le 3$, $0 \le x \le \frac{1}{2}y - \frac{1}{2}$ AND $3 \le y \le 5$, $0 \le x \le -\frac{1}{2}y + \frac{5}{2}$



Method ①

$$\iint_R xy \, dA$$

$$= \int_0^1 \int_{2x+1}^{-2x+5} xy \, dy \, dx$$

$$= \int_0^1 x\frac{y^2}{2} \Bigg|_{y=2x+1}^{-2x+5} dx =$$

$$= \frac{1}{2}\int_0^1 x(-2x+5)^2 - x(2x+1)^2 \, dx$$

$$= \frac{1}{2}\int_0^1 x(4x^2-20x+25) - x(4x^2+4x+1) \, dx$$

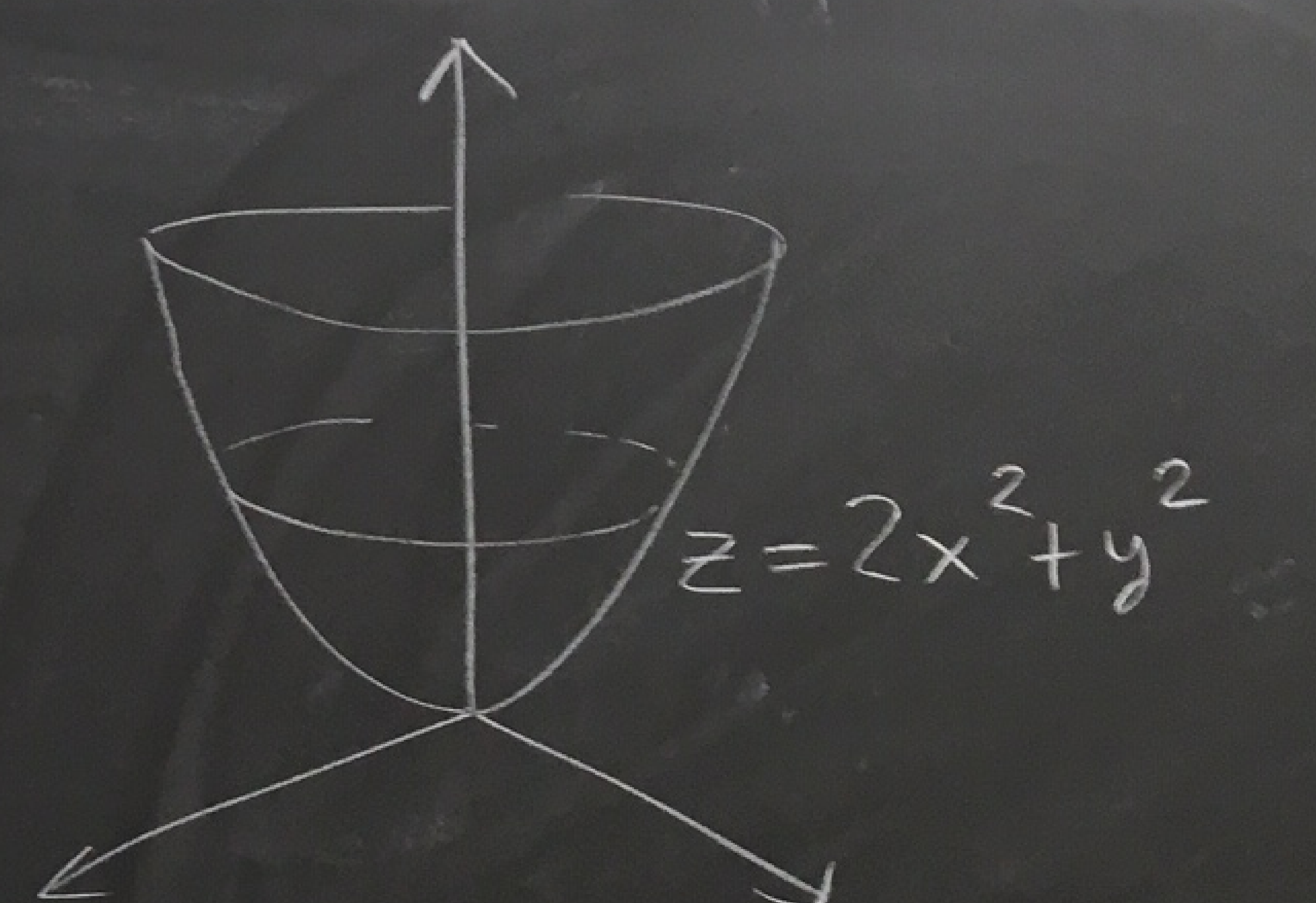
$$= \frac{1}{2}\int_0^1 (-24x^2+24x) \, dx$$

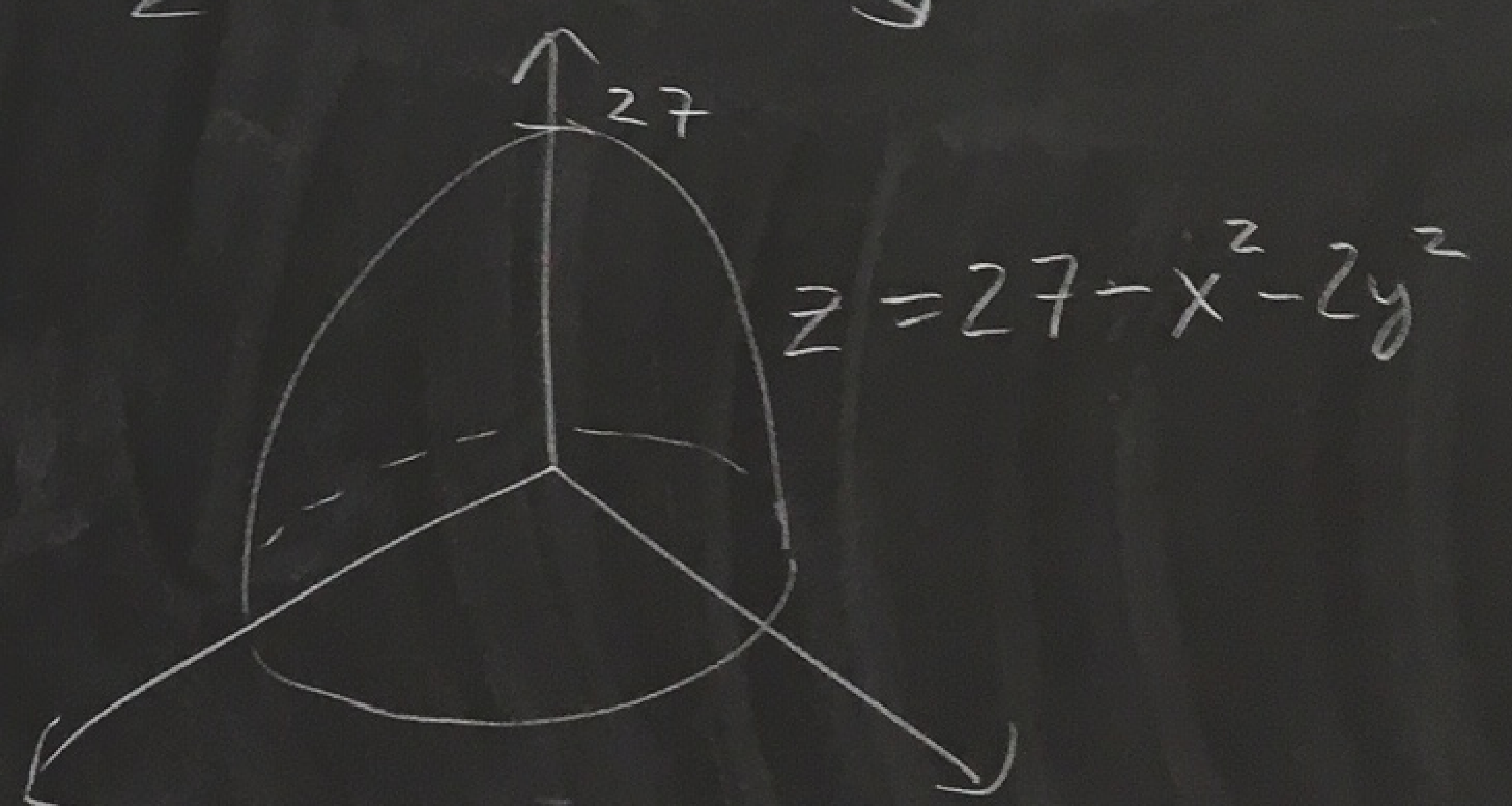
$$= \frac{1}{2}\left[-24\frac{x^3}{3}+24\frac{x^2}{2}\right]_0^1 = \frac{1}{2}\left[\frac{-24}{3}+\frac{24}{2}\right] = \frac{1}{2}\left(-8+12\right)$$

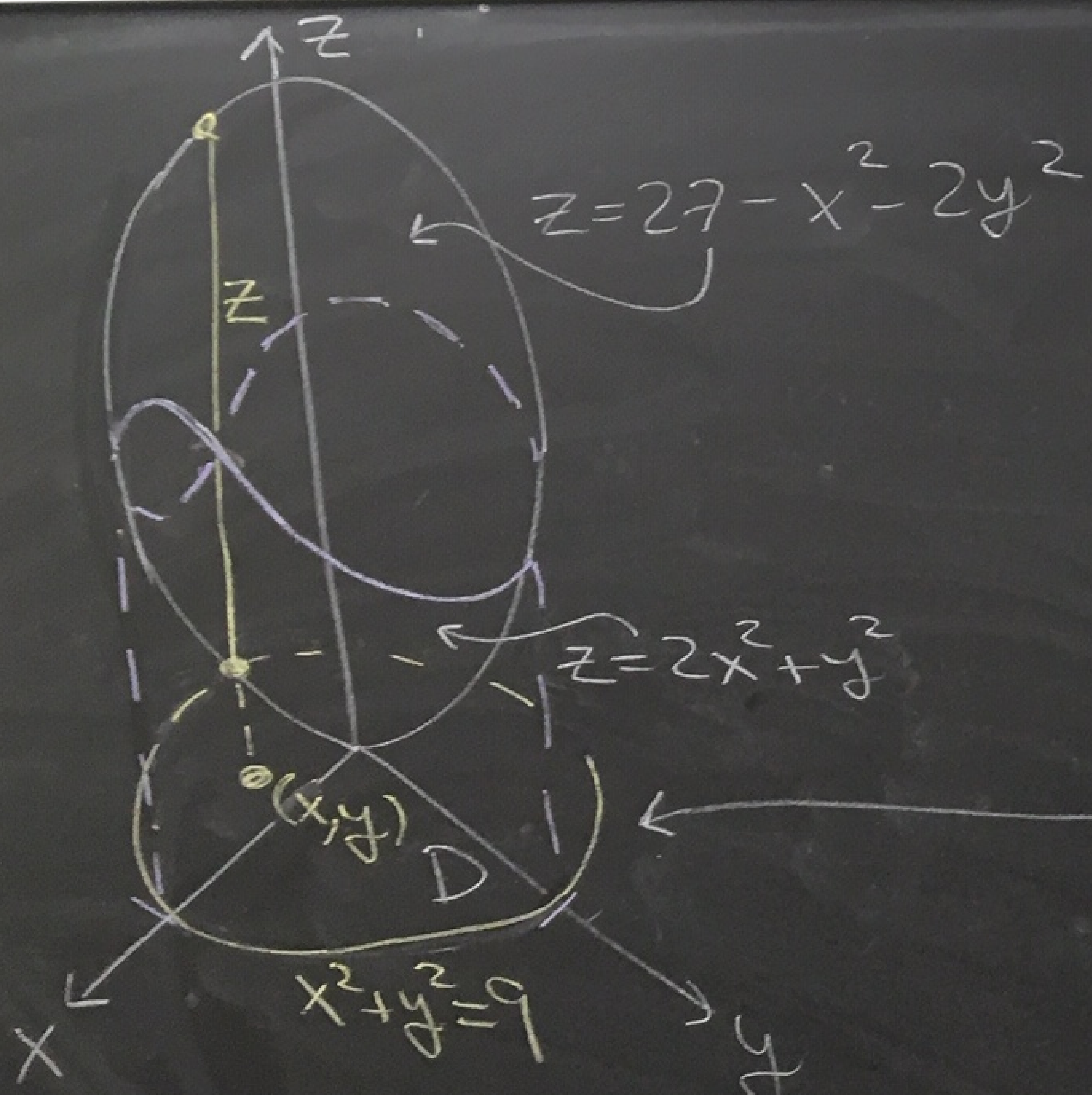
$$= 2$$

13.3

(20) Find the volume of the solid bounded by the paraboloids $z = 2x^2 + y^2$ and $z = 27 - x^2 - 2y^2$







Set them equal to each other

$$27-x^2-2y^2 = 2x^2+y^2$$

$$27 = 3x^2+3y^2$$

$$\boxed{9 = x^2+y^2}$$

$$\text{Volume} = \iint_D (27-x^2-2y^2)-(2x^2+y^2)\,dA$$

$$= \iint_D (27-3x^2-3y^2)\,dA$$

A

$$\boxed{\begin{array}{c} D \\ \hline 0 \le r \le 3 \\ 0 \le \theta \le 2\pi \end{array}}$$

$$\int_0^{2\pi}\int_0^3 (27-3r^2)\, r\, dr\, d\theta$$

$$27-3(x^2+y^2)=27-3r^2$$

$$\int_0^{2\pi}\int_0^3 27r-3r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} 27\frac{r^2}{2} - 3\frac{r^4}{4}\Big|_0^3 \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{27}{2}(3)^2 - \frac{3}{4}(3)^4\right] d\theta$$

$$= 243\int_0^{2\pi} \left(\frac{1}{2}-\frac{1}{4}\right) d\theta = 243\left(\frac{1}{4}\right)(2\pi) = \boxed{\frac{243}{2}\pi} = \boxed{\frac{3^5}{2}\pi}$$

$$\begin{array}{r} {}^{6}\ \ \\ 27 \\ 9 \\ \hline 243 \end{array}$$