Multi-colouring the Mycielskian of Graphs

Wensong Lin * Daphne Der-Fen Liu † Xuding Zhu ‡

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Abstract

A $k$-fold colouring of a graph is a function that assigns to each vertex a set of $k$ colours, so that the colour sets assigned to adjacent vertices are disjoint. The $k$-th chromatic number of a graph $G$, denoted by $\chi_k(G)$, is the minimum total number of colours used in a $k$-fold colouring of $G$. Let $\mu(G)$ denote the Mycielskian of $G$. For any positive integer $k$, it holds that $\chi_k(G) + 1 \leq \chi_k(\mu(G)) \leq \chi_k(G) + k$ [5]. Although both bounds are attainable, it was proved in [7] that if $k \geq 2$ and $\chi_k(G) \leq 3k - 2$, then the upper bound can be reduced by 1, i.e., $\chi_k(\mu(G)) \leq \chi_k(G) + k - 1$. We conjecture that for any $n \geq 3k - 1$, there is a graph $G$ with $\chi_k(G) = n$ and $\chi_k(\mu(G)) = n + k$. This is equivalent to conjecturing that the equality $\chi_k(\mu(K(n,k))) = n + k$ holds for Kneser graphs $K(n,k)$ with $n \geq 3k - 1$. We confirm this conjecture for $k = 2, 3$, or when $n$ is a multiple of $k$ or $n \geq 3k^2/\ln k$. Moreover, we determine the values of $\chi_k(\mu(C_{2q+1}))$ for $1 \leq k \leq q$.

*Department of Mathematics, Southeast University, Nanjing 210096, P.R. China. Supported in part by NSFC under grant 10671033. Email: wslin@seu.edu.cn.

†Corresponding Author. Department of Mathematics, California State University, Los Angeles, CA 90032, USA. Supported in part by the National Science Foundation under grant DMS 0302456. Email: dliu@calstatela.edu.

‡Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, and National Center for Theoretical Sciences, Taiwan. Supported in part by the National Science Council under grant NSC95-2115-M-110-013-MY3. Email: zhu@math.nsysu.edu.tw.
1 Introduction

In search of graphs with large chromatic number but small clique size, Mycielski [6] introduced the following construction: Let $G$ be a graph with vertex set $V$ and edge set $E$. Let $\overline{V}$ be a copy of $V$, $\overline{V} = \{ \overline{x} : x \in V \}$, and let $u$ be a new vertex. The Mycielskian of $G$, denoted by $\mu(G)$, is the graph with vertex set $V \cup \overline{V} \cup \{ u \}$ and edge set $E' = E \cup \{ x\overline{y} : xy \in E \} \cup \{ u\overline{x} : \overline{x} \in \overline{V} \}$. The vertex $u$ is called the root of $\mu(G)$; and for any $x \in V$, $\overline{x}$ is called the twin of $x$. For a graph $G$, denote $\chi(G)$ and $\omega(G)$, respectively, the chromatic number and the clique size of $G$. It is straightforward to verify that for any graph $G$ with $\omega(G) \geq 2$, we have $\omega(\mu(G)) = \omega(G)$ and $\chi(\mu(G)) = \chi(G) + 1$. Hence, one can obtain triangle free graphs with arbitrarily large chromatic number, by repeatedly applying the Mycielski construction to $K_2$.

Multiple-colouring of graphs was introduced by Stahl [10], and has been studied extensively in the literature. For any positive integers $n$ and $k$, we denote by $[n]$ the set $\{0, 1, \ldots, n-1\}$ and $\binom{[n]}{k}$ the set of all $k$-subsets of $[n]$. A $k$-fold $n$-colouring of a graph $G$ is a mapping, $f : V \rightarrow \binom{[n]}{k}$, such that for any edge $xy$ of $G$, $f(x) \cap f(y) = \emptyset$. In other words, a $k$-fold colouring assigns to each vertex a set of $k$ colours, where no colour is assigned to any adjacent vertices. Moreover, if all the colours assigned are from a set of $n$ colours, then it is a $k$-fold $n$-colouring. The $k$-th chromatic number of $G$ is defined as

$$\chi_k(G) = \min \{ n : G \text{ admits a } k \text{-fold } n\text{-colouring} \}.$$ 

The $k$-fold colouring is an extension of conventional vertex colouring. A 1-fold $n$-colouring of $G$ is simply a proper $n$-colouring of $G$, so $\chi_1(G) = \chi(G)$.

It is known [8] and easy to see that for any $k, k' \geq 1$, $\chi_{k+k'}(G) \leq \chi_k(G) + \chi_{k'}(G)$. This implies $\frac{\chi_k(G)}{k} \leq \chi(G)$. The fractional chromatic number of $G$ is
defined by
\[
\chi_f(G) = \inf \{ \frac{\chi_k(G)}{k} : k = 1, 2, \ldots \}.
\]

Thus \( \chi_f(G) \leq \chi(G) \) (cf. [8]).

For a graph \( G \), it is natural to ask the following two questions:

1. What is the relation between the fractional chromatic number of \( G \) and the fractional chromatic number of the Mycielskian of \( G \)?

2. What is the relation between the \( k \)-th chromatic number of \( G \) and the \( k \)-th chromatic number of the Mycielskian of \( G \)?

The first question was answered by Larsen, Propp and Ullman [4]. It turned out that the fractional chromatic number of \( \mu(G) \) is determined by the fractional chromatic number of \( G \): For any graph \( G \),

\[
\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}.
\]

The second question is largely open. Contrary to the answer of the first question in the above equality, the \( k \)-th chromatic number of \( \mu(G) \) is not determined by \( \chi_k(G) \). There are graphs \( G \) and \( G' \) with \( \chi_k(G) = \chi_k(G') \) but \( \chi_k(\mu(G)) \neq \chi_k(\mu(G')) \). So it is impossible to express \( \chi_k(\mu(G)) \) in terms of \( \chi_k(G) \). Hence, we aim at establishing sharp bounds for \( \chi_k(\mu(G)) \) in terms of \( \chi_k(G) \). Obviously, for any graph \( G \) and any positive integer \( k \), \( \chi_k(\mu(G)) \leq \chi_k(G) + k \). Combining this with a lower bound established in [5] we have:

\[
\chi_k(G) + 1 \leq \chi_k(\mu(G)) \leq \chi_k(G) + k. \quad (1)
\]

Moreover, it is proved in [5] that for any \( k \) both the upper and the lower bounds in (1) can be attained. On the other hand, it is proved in [7] that if \( \chi_k(G) \) is relatively small compared to \( k \), then the upper bound can be reduced.
Theorem 1 [7] If \( k \geq 2 \) and \( \chi_k(G) = n \leq 3k-2 \), then \( \chi_k(\mu(G)) \leq n+k-1 \).

In this article, we prove that for graphs \( G \) with \( \chi_k(G) \) relatively large compared to \( k \), then the upper bound in (1) cannot be improved. We conjecture that the condition \( n \leq 3k-2 \) in Theorem 1 is sharp.

Conjecture 1 If \( n \geq 3k-1 \), then there is a graph \( G \) with \( \chi_k(G) = n \) and \( \chi_k(\mu(G)) = n+k \).

A homomorphism from a graph \( G \) to a graph \( G' \) is a mapping \( f : V(G) \rightarrow V(G') \) such that \( f(x)f(y) \in E(G') \) whenever \( xy \in E(G) \). If \( f \) is a homomorphism from \( G \) to \( G' \) and \( c' \) is a \( k \)-fold \( n \)-colouring for \( G' \), then the mapping defined as \( c(x) = c'(f(x)) \) is a \( k \)-fold \( n \)-colouring of \( G \). Thus \( \chi_k(G) \leq \chi_k(G') \).

For positive integers \( n \geq 2k \), the Kneser graph \( K(n,k) \) has vertex set \( \binom{[n]}{k} \) in which \( x \sim y \) if \( x \cap y = \emptyset \). It follows from the definition that a graph \( G \) has a \( k \)-fold \( n \)-colouring if and only if there is a homomorphism from \( G \) to \( K(n,k) \). In particular, if \( k' = qk \) for some integer \( q \), then it is easy to show that \( \chi_{k'}(K(n,k)) = qn \). If \( k' \) is not a multiple of \( k \), then determining \( \chi_{k'}(K(n,k)) \) is usually a difficult problem. The well-known Kneser-Lovász Theorem [3] gives the answer to the case for \( k' = 1 \): \( \chi(K(n,k)) = n-2k+2 \). For \( k' \geq 2 \), the values of \( \chi_{k'}(K(n,k)) \) are still widely open.

Notice that, a homomorphism from \( G \) to \( G' \) induces a homomorphism from \( \mu(G) \) to \( \mu(G') \). Hence, we have

\[
\max\{\chi_k(\mu(G)) : \chi_k(G) = n\} = \chi_k(\mu(K(n,k))).
\]

Therefore, Conjecture 1 is equivalent to

Conjecture 2 If \( n \geq 3k-1 \), then \( \chi_k(\mu(K(n,k))) = n+k \).

In this paper, we confirm Conjecture 2 for the following cases:
• $n$ is a multiple of $k$ (Section 2),
• $n \geq 3k^2/\ln k$ (Section 2),
• $k \leq 3$ (Section 3).

It was proved in [5] that the lower bound in (1) is sharp for complete graphs $K_n$ with $k \leq n$. That is, if $k \leq n$, then $\chi_k(K_n) = \chi_k(K_n) + 1 = kn + 1$. In Section 4, we generalize this result to circular complete graphs $K_{p/q}$ (Corollary 10). Also included in Section 4 are complete solutions of the $k$-th chromatic number for the Mycielskian of odd cycles $C_{2q+1}$ with $k \leq q$.

2 Kneser graphs with large order

In this section, we prove for any $k$, if $n = qk$ for some integer $q \geq 3$ or $n \geq 3k^2/\ln k$, then $\chi_k(\mu(K(n,k))) = n + k$.

In the following, the vertex set of $K(n,k)$ is denoted by $V$. The Mycielskian $\mu(K(n,k))$ has the vertex set $V \cup \overline{V} \cup \{u\}$. For two integers $a \leq b$, let $[a,b]$ denote the set of integers $i$ with $a \leq i \leq b$.

Lemma 2 For any positive integer $k$, $\chi_k(\mu(K(3k,k))) = 4k$.

Proof. Suppose to the contrary, $\chi_k(\mu(K(3k,k))) \leq 4k - 1$. Let $c$ be a $k$-fold colouring of $\mu(K(3k,k))$ using colours from the set $[0, 4k - 2]$. Without loss of generality, assume $c(u) = [0, k - 1]$. Let $X = \{x \in V : c(x) \cap c(u) = \emptyset\}$. Then $X$ is an independent set in $K(3k,k)$; for if $v, w \in X$ and $v \sim w$, then $v, w$ have a common neighbor, say $\overline{x}$, in $\overline{V}$, implying that $c(v), c(w), c(\overline{x})$ and $c(u)$ are pairwise disjoint. So $|c(u)| + |c(\overline{x})| + |c(v)| + |c(w)| = 4k$, a contradiction. Hence, the vertices of $V$ can be partitioned into $k + 1$ independent sets: $X$ and $A_i = \{v \in V : i = \min c(v)\}$, $i = 0, 1, \ldots, k - 1$, contradicting the fact that $\chi(K(3k,k)) = k + 2$. 

\[ \blacksquare \]
Lemma 3  For any \( n \geq 3k - 1 \),

\[
\chi_k(\mu(K(n,k))) \geq \chi_k(\mu(K(n-k,k))) + k.
\]

Proof. Suppose \( \chi_k(\mu(K(n,k))) = m \). Let \( c \) be a \( k \)-fold colouring for \( \mu(K(n,k)) \) using colours from \([0, m-1]\). Assume \( c(u) = [0, k-1] \). Since \( \chi(K(n,k)) = n - 2k + 2 > k \), there exists some vertex \( v \) in \( V \) with \( c(v) \cap [0, k-1] = \emptyset \). Without loss of generality, assume \( c(v) = [k, 2k-1] \). Let \( N \) be the set of neighbors of \( v \) in \( V \), and let \( N' = \{ w \in V : w \in N \} \).

Then the subgraph of \( \mu(K(n,k)) \) induced by \( N \cup N' \cup \{ u \} \) is isomorphic to \( \mu(K(n-k,k)) \). Denote this subgraph by \( G' \). The colouring \( c \) restricted to \( G' \) is a \( k \)-fold colouring using colours from \([0, m-1]\) \( \setminus [k, 2k-1] \), which implies \( \chi_k(G') = \chi_k(\mu(K(n-k,k))) \leq m - k \).

\[\square\]

Corollary 4  For any integers \( q \geq 3 \) and \( k \geq 1 \), \( \chi_k(\mu(K(qk,k))) = (q+1)k \).

Next we prove that \( \chi_k(\mu(K(n,k))) = n+k \) holds for \( n \geq 3k^2/\ln k \). It was proved by Hilton and Milner [2] that if \( X \) is an independent set of \( K(n,k) \) and \( \cap_{x \in X} x = \emptyset \), then \( |X| \leq 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1} \).

For any positive integer \( k \), let \( \phi(k) \) be the minimum \( n \) such that

\[
\frac{n ((n-k-1)(n-k-2) \ldots (n-2k+1) - (k-1)!)}{k(n-1)(n-2) \ldots (n-k+1)} > 1.  \tag{2}
\]

Theorem 5 Let \( n \) and \( k \) be integers with \( n \geq \phi(k) \). Then

\[
\chi_k(\mu(K(n-1,k))) \leq \chi_k(\mu(K(n,k))) - 1.
\]

Proof. Let \( t = \chi_k(\mu(K(n,k))) \) and let \( c \) be a \( k \)-fold \( t \)-colouring of \( \mu(K(n,k)) \) using colours from \([0, t-1]\). Assume \( c(u) = [0, k-1] \). For \( i \in [0, t-1] \), let \( S_i = \{ x \in V : i \in c(x) \} \). Then \( \sum_{i=0}^{t-1} |S_i| = k\binom{n}{k} \), since each vertex appears in exactly \( k \) of the \( S_i \)'s.
Since \( t \leq n + k \), by a straightforward calculation, inequality (2) implies that
\[
k\left(\begin{array}{c}n \\ k\end{array}\right) > (t - k) \left(1 + \left(\begin{array}{c}n - 1 \\ k - 1\end{array}\right) - \left(\begin{array}{c}n - k - 1 \\ k - 1\end{array}\right)\right) + k\left(\begin{array}{c}n - 1 \\ k - 1\end{array}\right).
\]
Therefore, at least \( k + 1 \) of the \( S_i \)'s satisfy the following:
\[
|S_i| > 1 + \left(\begin{array}{c}n - 1 \\ k - 1\end{array}\right) - \left(\begin{array}{c}n - k - 1 \\ k - 1\end{array}\right).
\]
Hence there exists \( i^* \not\in [0, k - 1] \) with \( |S_{i^*}| > 1 + \left(\begin{array}{c}n - 1 \\ k - 1\end{array}\right) - \left(\begin{array}{c}n - k - 1 \\ k - 1\end{array}\right) \). This implies \( \cap_{x \in S_{i^*}} x \neq \emptyset \). Note that the intersection \( \cap_{x \in S_{i^*}} x \) contains only one integer.

For otherwise, assume \( a \in W = \cap_{x \in S_{i^*}} x \) and \( W \setminus \{a\} \neq \emptyset \). Let \( x' \) be a vertex containing \( a \), and \( y' \) be a vertex such that \( y' \cap W = W \setminus \{a\} \) and \( y' \cap x' \neq \emptyset \). Then \( S' = S_{i^*} \cup \{x', y'\} \) is an independent set with \( |S'| > 1 + \left(\begin{array}{c}n - 1 \\ k - 1\end{array}\right) - \left(\begin{array}{c}n - k - 1 \\ k - 1\end{array}\right) \) and \( \cap_{x \in S'} x = \emptyset \), a contradiction.

Assume \( \cap_{x \in S_{i^*}} x = \{a\} \). If \( y \in K(n, k) \) and \( y \) intersects every \( x \in S_{i^*} \), then \( a \in y \). For otherwise, \( S' = S_{i^*} \cup \{y\} \) is an independent set with \( S_{i^*} \subset S' \) and \( \cap_{x \in S'} x = \emptyset \), a contradiction. We conclude that for any \( y \in K(n, k) \), if \( a \not\in y \), then none of \( S_{i^*} \cup \{y\} \) and \( S_{i^*} \cup \{7\} \) is an independent set in \( \mu(K(n, k)) \), which implies that \( i^* \not\in c(y) \) and \( i^* \not\in c(7) \).

By letting \( a = n \), the restriction of \( c \) to the subgraph \( \mu(K(n - 1, k)) \) gives a \( k \)-fold \((t - 1)\)-colouring of \( \mu(K(n - 1, k)) \).

**Corollary 6** For any \( n \geq \max\{2k + 1, N\} \), \( \chi_k(\mu(K(n, k))) = n + k \), where \( N \) is defined as follows. If \( \phi(k) = qk + 1 \), then \( N = qk \); otherwise, \( N \) is the smallest integer such that \( N \) is a multiple of \( k \) and \( N \leq \phi(k) \).

**Proof.** By Corollary 4, \( \chi_k(\mu(K(N, k))) = N + k \). By Theorem 5,
\[
\chi_k(\mu(K(n, k))) \geq (n - N) + \chi_k(\mu(K(N, k))) = n + k.
\]
Although it might be hard to find a simple formula for the function $\phi(k)$ defined in the above, one can easily learn that $\phi(k)$ has order $k^2/\ln k$.

**Corollary 7** If $k \geq 4$ and $n \geq 3k^2/\ln k$, then $\chi_k(\mu(K(n,k))) = n + k$.

**Proof.** Assume $n \geq 3k^2/\ln k$. Then
\[
\frac{n[(n-k-1)(n-k-2)\ldots(n-2k+1)-(k-1)!]}{k(n-1)(n-2)\ldots(n-k+1)} > \frac{(n-1)(n-k-1)(n-k-2)\ldots(n-2k+1)}{k(n-1)(n-2)\ldots(n-k+1)}
\]
\[
> \frac{n-1}{k} \left( \frac{n-2k}{n-k} \right)^{k-1}
\]
\[
> \frac{n-1}{k} e^{-k(k-1)/(n-2k)}
\]
\[
> \frac{2k}{\ln k} e^{-k(k-1)(\ln k)/2k^2}
\]
\[
> \frac{2k}{\sqrt{k \ln k}} > 1.
\]

Therefore, $n \geq N$ for the $N$ defined in Corollary 6, so the result follows. ■

In Corollary 7, $3k^2/\ln k$ can be replaced by $(1 + \epsilon)k^2/\ln k$ for any $\epsilon > 0$, provided that $k$ is large enough.

3 **$K(n, 2)$ and $K(n, 3)$**

In this section, we confirm Conjecture 2 for $k \leq 3$. The case $k = 1$ was proved by Mycielski. For $k = 2, 3$, the value of $\phi(k)$ defined in (2) in Section 2 can be easily determined: $\phi(2) = 6$ and $\phi(3) = 10$. Thus to prove Conjecture 2 for $k = 2, 3$, by Corollary 6 it suffices to show that $\chi_2(\mu(K(5,2))) = 7$ and $\chi_3(K(8,3)) = 11$. As it was proved in [5] that $\chi_2(\mu(K(5,2))) = 7$, the case $k = 2$ is settled.
In the following, we confirm the case $k = 3$.

**Theorem 8** $\chi_3(\mu(K(8, 3))) = 11$.

**Proof.** As $\chi_k(K(8, 3)) \leq 11$, it suffices to show $\chi_k(K(8, 3)) > 10$. Assume to the contrary, there exists a 3-fold 10-colouring $c$ of $\mu(K(8, 3))$, using colours from the set $\{a_0, a_1, \ldots, a_9\}$. For simplicity, we denote each vertex in $V$ by $(ijk)$, where $i, j, k \in \{0, 1, 2, \ldots, 7\}$, and its twin by $(\overline{ijk})$; and for $s \leq t$, we denote the set of colours $\{a_s, a_{s+1}, \ldots, a_t\}$ by $a[s, t]$.

Assume $c(u) = a[0, 2]$. Let $X = \{x \in V : c(x) \cap c(u) = \emptyset\}$. For $x \in X$ and $i \not\in x$, let $M_i(x) = \{v \in V : v \setminus x = \{i\}\}$. For a set $A$ of vertices, let $c\langle A \rangle = \bigcup_{x \in A} c(x)$.

**Claim 1** For any $x \in X$, there is at most one integer $i \not\in x$ for which $c\langle M_i(x) \rangle \not\subseteq c(x) \cup c(u)$.

**Proof.** Assume the claim is not true. Without loss of generality, assume that $x = (012)$, $c(x) = a[3, 5]$ and $c(M_3(x))$, $c(M_7(x)) \not\subseteq c(x) \cup c(u) = a[0, 5]$. We may assume $a_6 \in c(M_3(x))$ and $a_t \in c(M_7(x))$ for some $t \in [6, 9]$. For any $i, j, k \in [4, 7]$, $(ijk) \sim x, u, M_3(x)$. Hence $c(\overline{ijk}) = a[7, 9]$. Similarly, for any $i, j, k \in [3, 6]$, $c(\overline{ijk}) = a[6, 9] - \{a_t\}$. As $c(456) = a[7, 9] = a[6, 9] - \{a_t\}$, we conclude that $t = 6$.

Let $W := \{(034), (157), (026), (134), (257)\}$. Every vertex in $W$ is adjacent to some $(ijk)$, with $i, j, k \in [4, 7]$ or $i, j, k \in [3, 6]$. Hence, $c(W) \subseteq a[0, 6]$. This is impossible, as $W$ induces a $C_5$ while it is known [10] that $\chi_3(C_5) = 8$.

**Claim 2** Let $x, y \in X$. If $x \neq y$, then $c(x) \neq c(y)$. Moreover, if $x \cap y \neq \emptyset$, then $|c(x) \cap c(y)| = 2$.  

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Proof. Let \(x, y \in X\), \(x \neq y\). Assume to the contrary, \(c(x) = c(y)\). Then \(x \cap y \neq \emptyset\). Assume \(|x \cap y| = 2\), say \(x = (012), y = (013) \in X\) and \(c(y) = c(x) = a[3, 5]\). Then \(c(245), c(367) \subseteq a[6, 9]\), implying \(|c(245) \cap a[0, 2]| \geq 2\) and \(|c(367) \cap a[0, 2]| \geq 2\). This is impossible as \((367) \sim (245)\).

Next, assume \(|x \cap y| = 1\), say \(x = (012), y = (234)\) and \(c(x) = c(y) = a[3, 5]\). By Claim 1, there exists \(i \in \{5, 6, 7\}\), say \(i = 5\), \(c(M_i(x)) \subseteq c(x) \cup c(u) = a[0, 5]\). Hence \(c(015) = a[0, 2]\) (as \(015 \sim (234)\)). Then \(c(346), c(015) \subseteq a[6, 9]\), a contradiction, as \(345 \sim 015\). Hence, \(c(x) \neq c(y)\).

To prove the moreover part, assume \(x \cap y \neq \emptyset\). Then there is some \(z \in V\) with \(z \sim x, y\). Thus \(c(x) \cup c(y) \cup c(u)\) is disjoint from \(c(\pi)\). This implies \(|c(x) \cap c(y)| = 2\).

In the remainder of the proof, we use Schrijver graphs. For \(n \geq k\), the Schrijver graph, denoted by \(S(n, k)\), is a subgraph of \(K(n, k)\) induced by the vertices that do not contain any pair of consecutive integers in the cyclic order of \([n]\). Schrijver [9] proved that \(\chi(K(n, k)) = \chi(S(n, k))\) and \(S(n, k)\) is vertex critical.

Denote the subgraph of \(K(8, 3)\) induced by \(V - X\) by \(K(8, 3) \setminus X\). Then \(K(8, 3) \setminus X\) has a 3-vertex-colouring \(f\), defined by \(f(v) = \min\{c(v)\}\). Hence, \(S(8, 3)\) can not be a subgraph of \(K(8, 3) \setminus X\). In what follows, we frequently use the fact that if, for some ordering of \(\{0, 1, \ldots, 7\}\), each vertex \(x \in X\) contains a pair of cyclically consecutive integers in \(\{0, 1, \ldots, 7\}\), then \(K(8, 3) \setminus X\) contains \(S(8, 3)\) as a subgraph, which is a contradiction.

Claim 3 For any \(x, y \in X\), \(x \cap y \neq \emptyset\).

Proof. Assume to the contrary, \(x = (012), y = (567) \in X\). Suppose there is a vertex \(z \in X \setminus \{x, y\}\) which intersects both \(x, y\). By Claim 2, \(|c(z) \cap c(x)| = 2\) and \(|c(z) \cap c(y)| = 2\), which is a contradiction, as \(c(x) \cap c(y) = \emptyset\). Therefore,
any \( z \in X \setminus \{x, y\} \) is either disjoint from \( x \) or disjoint from \( y \). We partition \( X \) into two sets, \( A_x \) and \( A_y \), that include vertices disjoint from \( x \) or from \( y \), respectively.

Next we claim \( A_x = \{y\} \) or \( A_y = \{x\} \). For each \( z \in A_y \), applying the above discussion on \( x \) and \( y \) to \( z \) and \( y \), one can show that for any \( z' \in A_x \), \( z \cap z' = \emptyset \). Hence, if \( A_y - \{x\} \neq \emptyset \) and \( A_x - \{y\} \neq \emptyset \), then we may assume \( z \subseteq [0, 3] \) for all \( z \in A_y \), and \( z' \subseteq [4, 7] \) for all \( z' \in A_x \). This implies that every vertex of \( X \) contains two consecutive integers. Thus, \( A_x = \{y\} \) or \( A_y = \{x\} \).

Assume \( A_x = \{y\} \). If \((024) \notin X\), then clearly every vertex of \( X \) contains two consecutive integers. Suppose \( z_1 = (024) \in X \). If \((023) \notin X\), then by exchanging 3 and 4 in the cyclic ordering, every vertex in \( X \) contains two consecutive integers. Assume \( z_2 = (023) \in X \). By Claim 1, for some \( i \in \{1, 2\} \), \( c(z_i) \subseteq c(x) \cup c(u) \), and hence \( c(x) = c(z_i) \) (since \( z_i \in X \)), contradicting Claim 2.

It follows from Claims 2 and 3 that for any distinct \( x, y \in X \), \(|c(x) \cap c(y)| = 2 \). There are at most five 3-subsets of \( a[3, 9] \) that pairwise have two elements in common. Thus \(|X| \leq 5 \). By Claim 3, it is straightforward to verify that there exists an ordering of \( \{0, 1, 2, \ldots, 7\} \) such that each \( x \in X \) contains a pair of cyclic consecutive integers. The details are omitted, as they are a bit tedious yet apparent.

## 4 Circular cliques and odd cycles

For any positive integers \( p \geq 2q \), the circular complete graph (or circular clique) \( K_{p/q} \) has vertex set \([p]\) in which \( ij \) is an edge if and only if \( q \leq |i - j| \leq p - q \). Circular cliques play an essential role in the study of circular chromatic number of graphs (cf. [12, 13]). A homomorphism from \( G \) to \( K_{p/q} \) is also called a \((p,q)\)-colouring of \( G \). The circular chromatic number of \( G \) is
defined as

\[ \chi_c(G) = \inf\{p/q : G \text{ has a } (p,q)\text{-colouring}\}. \]

It is known [12] that for any graph \( G \), \( \chi_f(G) \leq \chi_c(G) \). Moreover, a result in [1] implies that if \( \chi_f(G) = \chi_c(G) \) then for any positive integer \( k \),

\[ \chi_k(G) = \lceil k\chi_f(G) \rceil. \]

As \( \chi_c(K_{p/q}) = \chi_f(K_{p/q}) = p/q \), we have

\[ \chi_k(K_{p/q}) = \lceil kp/q \rceil. \]

Let \( m = \lceil kp/q \rceil \). Indeed, a \( k \)-fold \( m \)-colouring \( c \) of \( K_{p/q} \), using colours \( a_0, a_1, \ldots, a_{m-1} \), can be easily constructed as follows. For \( j = 0, 1, \ldots, m-1 \), assign colour \( a_j \) to vertices \( jq, jq+1, \ldots, (j+1)q-1 \). Here the calculations are modulo \( p \). Observe that \( c \) is a \( k \)-fold colouring for \( K_{p/q} \), because each colour \( a_j \) is assigned to an independent set of \( K_{p/q} \), and the union \( \bigcup_{j=0}^{m-1} \{jq, jq+1, \ldots, (j+1)q-1\} = [0, mq-1] \) is an interval of \( mq \) consecutive integers. As \( mq \geq kp \), for each integer \( i \), there are at least \( k \) integers \( t \in [0, mq-1] \) that are congruent to \( i \) modulo \( p \), i.e., there are at least \( k \) colours assigned to each vertex \( i \) of \( K_{p/q} \). (Here, for convenience, we modify the definition of a \( k \)-fold colouring to be a colouring which assigns to each vertex a set of at least \( k \) colours.)

Now we extend the above \( k \)-fold colouring \( c \) of \( K_{p/q} \) to a \( k \)-fold colouring for \( \mu(K_{p/q}) \) by assigning at least \( k \) colours to each vertex in \( V \cup \{u\} \). Let \( S = a[m-k, m-1] \) and let \( c(u) = S \). For \( i \in V(K_{p/q}) \), let \( g(\overline{i}) = c(i) \setminus S \). Then \( |g(\overline{i})| \) is equal to the number of integers in the interval \([0, (m-k)q-1]\) that are congruent to \( i \) modulo \( p \). Hence \( |g(\overline{i})| \geq [(m-k)q/p] \). Let \( b = k - \lceil (m-k)q/p \rceil \), and let \( c(\overline{i}) = g(\overline{i}) \cup \{a_m, a_{m+1}, \ldots, a_{m+b-1}\} \). Then \( c \) is a \( k \)-fold \( (m+b) \)-colouring of \( \mu(K_{p/q}) \), implying \( \chi_k(\mu(K_{p/q})) \leq m+b \).
Theorem 9 Suppose $p, q, k$ are positive integers with $p \geq 2q$. Then

$$\lceil kp/q + kq/p \rceil \leq \chi_k(\mu(K_{p/q})) \leq \lceil kp/q \rceil + \lceil kq/p \rceil.$$  

Proof. The lower bound follows from the result that $\chi_f(\mu(K_{p/q})) = \chi_f(K_{p/q}) + \frac{1}{\chi_f(K_{p/q})} = \frac{p}{q} + \frac{q}{p}$. For the upper bound, we have shown in the previous paragraph that $\chi_k(\mu(K_{p/q})) \leq m + b$, where $m = \lceil kq/p \rceil$ and $b = k - \lceil (m - k)q/p \rceil$. By letting $m = (kp + s)/q$, easy calculation shows that $b = \lceil (kq - s)/p \rceil \leq \lceil kq/p \rceil$.

It was proved in [5] that $\chi_k(\mu(K_n)) = \chi_k(K_n) + 1 = kn + 1$ holds for $k \leq n$. By Theorem 9, this result can be generalized to circular cliques.

Corollary 10 If $k \leq p/q$, then $\chi_k(\mu(K_{p/q})) = \chi_k(K_{p/q}) + 1$.

Proof. As $\chi_k(\mu(G)) \geq \chi_k(G) + 1$ holds for any graph $G$, it suffices to note that when $k \leq p/q$, Theorem 9 implies that $\chi_k(\mu(K_{p/q})) \leq \chi_k(K_{p/q}) + 1$.

Corollary 11 If $k = tq$ is a multiple of $q$, then $\chi_k(\mu(K_{p/q})) = tp + \lceil kq/p \rceil$; if $k = sp$ is a multiple of $p$, then $\chi_k(\mu(K_{p/q})) = sq + \lceil kp/q \rceil$.

Corollary 11 implies that for any integer $s$ with $1 \leq s \leq \lceil k/2 \rceil$, there is a graph $G$ with $\chi_k(\mu(G)) = \chi_k(G) + s$.

If $p = 2q + 1$, then $K_{p/q}$ is the odd cycle $C_{2q+1}$. Assume $k \leq q$. By Theorem 9,

$$2k + \lceil (k + 1)/2 \rceil \leq \chi_k(\mu(C_{2q+1})) \leq 2k + \lceil (k + 2)/2 \rceil.$$  

In particular, if $k$ is even, then $\chi_k(\mu(C_{2q+1})) = 5k/2 + 1$; if $k$ is odd, then $\chi_k(\mu(C_{2q+1})) \in \{2k + \frac{k+1}{2}, 2k + \frac{k+3}{2}\}$. It was proved in [5] that $\chi_k(\mu(C_{2q+1})) = 2k + \frac{k+3}{2}$ if $k$ is odd and $k \leq q \leq \frac{3k-1}{2}$. In the next theorem, we completely determine the value of $\chi_k(\mu(C_{2q+1}))$ for $3 \leq k \leq q$.  

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Theorem 12 Let $k$ be an odd integer, $k \geq 3$. Then
\[
\chi_k(\mu(C_{2q+1})) = \begin{cases} 
2k + \frac{k+3}{2}, & \text{if } k \leq q \leq \frac{3k+3}{2}; \\
2k + \frac{k+1}{2}, & \text{if } q \geq \frac{3k+5}{2}.
\end{cases}
\]

Proof. Denote $V(C_{2q+1}) = \{v_0, v_1, \ldots, v_{2q}\}$, where $v_i \sim v_{i+1}$. Throughout the proof, all the subindices are taken modulo $2q + 1$.

We first consider the case $k \leq q \leq \frac{3k+3}{2}$. Assume to the contrary, \( \chi_k(\mu(C_{2q+1})) = 2k + \frac{k+1}{2} \). Let $c$ be a $k$-fold colouring of $\mu(C_{2q+1})$ using colours from the set $a[0, 2k + \frac{k-1}{2}]$. Without loss of generality, assume $c(u) = a[0, k - 1]$.

Denote by $X$ the colour set $a[k, 2k + \frac{k-1}{2}]$. For $i = 0, 1, \ldots, 2q$, let $W_i = c(v_i)$, $X_i = W_i \cap X$, and $Y_i = W_i \cap a[0, k - 1]$. Then $W_i = Y_i \cup X_i$ and $|X_i| + |Y_i| = k$. For each $i$, since $c(v_{i-1}) \subseteq X$ and $(c(v_{i-1}) \cup c(v_i)) \cap c(v_i) = \emptyset$, we have $|X_{i-1} \cup X_{i+1}| \leq |X| - k = (k + 1)/2$. As $|W_i \cup W_{i+1}| = 2k$, we have $|X_i \cup X_{i+1}| \geq k$. Hence, for each $i$, $\frac{k-1}{2} \leq |X_i| \leq \frac{k+1}{2}$.

Partition $V = \{v_0, v_1, \ldots, v_{2q}\}$ into the following two sets:

\[
\begin{align*}
A_1 &= \{v_i \in V : |X_i| = \frac{k-1}{2}\}, \\
A_2 &= \{v_i \in V : |X_i| = \frac{k+1}{2}\}.
\end{align*}
\]

Observation A. All the following hold for every $i \in [0, 2q]$:

1. If $v_i \in A_1$, then $v_{i-1}, v_{i+1} \in A_2$.

2. If $v_i, v_{i+2} \in A_2$, then $X_i = X_{i+2}$; if $v_i, v_{i+2} \in A_1$, then $|X_i \setminus X_{i+2}| \leq 1$ and $|X_{i+2} \setminus X_i| \leq 1$.

3. Assume $v_i \in A_1$ for some $i$. If $v_{i+2} \in A_2$ (or $v_{i-2} \in A_2$, respectively), then $X_i \subseteq X_{i+2}$ (or $X_i \subseteq X_{i-2}$, respectively).

For each $i$, as $|X_i| + |Y_i| = k$, one has $\frac{k-1}{2} \leq |Y_i| \leq \frac{k+1}{2}$. Similar to the above discussion on $X_i$'s, we have:

Observation B. The following hold for all $i \in [0, 2q]$:
1. If \( v_i, v_{i+2} \in A_1 \), then \( Y_i = Y_{i+2} \).

2. Assume \( v_i \in A_1 \) for some \( i \). If \( v_{i+2} \in A_2 \) (or \( v_{i-2} \in A_2 \), respectively), then \( Y_{i+2} \subseteq Y_i \) (or \( Y_{i-2} \subseteq Y_i \), respectively).

3. Assume \( v_i, v_{i+2} \in A_2 \) for some \( i \). If \( v_{i+1} \in A_1 \), then \( Y_{i+1} \subseteq Y_i \); if \( v_{i+1} \in A_2 \), then \( |Y_{i+2} \setminus Y_i| \leq 1 \) and \( |Y_i \setminus Y_{i+2}| \leq 1 \).

By Observation A (1), there exists some \( i \) such that \( v_i, v_{i+1} \in A_2 \). Without loss of generality, assume \( v_0, v_1 \in A_2 \).

**Claim 1.** \( |A_1| = k + 2 \). Moreover, all the following hold:

1. \( \bigcup_{i=0}^{2q} X_i = X_0 \cup X_1 \cup \{w^*\} \) for some \( w^* \not\in X_0 \cup X_1 \).

2. For each \( v_i \in A_1 \), \( i \in [0, 2q] \), there exists some \( x \in X_{i-2} \setminus X_i \). In addition, if \( x \neq w^* \), then \( x \in X_0 \) if \( i \) is even; and \( x \in X_1 \) if \( i \) is odd.

3. For each \( x \in X_0 \cup X_1 \cup \{w^*\} \) there exists a unique \( i \in [0, 2q] \) such that \( x \in X_i \setminus X_{i+2} \). In addition,
   - if \( x = w^* \), then \( x \not\in X_{i+2} \cup X_{i+3} \cup \ldots \cup X_{2q} \);
   - if \( x \in X_0 \), then \( i \) is even and \( x \not\in X_{i+2} \cup X_{i+4} \cup \ldots \cup X_{2q} \); and
   - if \( x \in X_1 \), then \( i \) is odd and \( x \not\in X_{i+2} \cup X_{i+4} \cup \ldots \cup X_{2q-1} \).

**Proof.** Consider the sequence \( (X_0, X_2, \ldots, X_{2q}, X_1) \). Because \( X_0 \cap X_1 = \emptyset \), for each \( x \in X_0 \), there exists some even number \( i \in [0, 2q] \) such that \( x \in X_i \setminus X_{i+2} \). By Observation A, \( X_i \setminus X_{i+2} = \{x\} \) and \( v_{i+2} \in A_1 \). Since \( |X_0| = \frac{k+1}{2} \), we conclude that there exist \( \frac{k+1}{2} \) even integers \( i \in [0, 2q] \) with \( |X_i \setminus X_{i+2}| = 1 \) and \( v_{i+2} \in A_1 \). Similarly, by considering the sequence \( (X_1, X_3, \ldots, X_{2q-1}, X_0) \), there exist \( \frac{k+1}{2} \) odd integers \( i \in [0, 2q] \) with \( |X_i \setminus X_{i+2}| = 1 \) and \( v_{i+2} \in A_1 \). Hence, \( |A_1| \geq k + 1 \).
Let $i^*$ be the smallest nonnegative integer such that $|X_{i^*+2} \setminus X_{i^*}| = 1$. Note, by the above discussion, $i^*$ exists. Let $X_{i^*+2} \setminus X_{i^*} = \{w^*\}$. It can be seen that $w^* \not\in X_0 \cup X_1$. By the same argument as in the previous paragraph (using either the even or the odd sequence depending on the parity of $i^*$), there exists some $i \geq i^*$ such that $w^* \in X_i \setminus X_{i+2}$ and $v_{i+2} \in A_1$. Moreover, this $i$ is different from the $i$'s observed in the previous paragraph. So, $|A_1| \geq k + 2$.

By a similar discussion applied to $Y_0$ and $Y_1$ one can show that there are at least $k$ integers $i$ such that $|Y_i \setminus Y_{i+2}| = 1$.

Combining all the above discussion, to complete the proof (including the moreover part) it is enough to show $|A_1| \leq k+2$. Consider a sequence $v_i, v_{i+1}, \ldots, v_{i+s}, v_{i+s+1}$ with $v_i, v_{i+s+1} \in A_1$ and $v_{i+1}, \ldots, v_{i+s} \in A_2$. Then $s > 0$ holds, and by Observation B, there are at most $s – 1$ integers $j$ in $[i, i+s]$ such that $|Y_j \setminus Y_{j+2}| = 1$. Hence, there are at most $|A_2| – |A_1|$ integers $i$ in $[0, 2q]$ with $|Y_i \setminus Y_{i+2}| = 1$. This implies, by the previous paragraph, $|A_2| – |A_1| \geq k$. Recall, $|A_2| + |A_1| = 2q + 1 \leq 3k + 4$. Therefore, $|A_1| \leq k + 2$. **Claim 2.** For any $v_i, v_j \in A_1$ with $i \neq j$, we have $|i – j| \geq 3$.

**Proof.** Suppose the claim fails. Without loss of generality, by Observation A (1), we may assume there exists some $i \in [0, 2q]$ such that $v_{i-1}, v_{i+1} \in A_1$ and $v_{i-3}, v_{i-2}, v_i, v_{i+2} \in A_2$. By Observation A (2), $X_{i-2} = X_i = X_{i+2}$. Assume $i$ is odd. (The proof for $i$ even is symmetric.) By Claim 1 (2), there exist $w_1 \in X_{i-3} \setminus X_{i-1}$ and $w_2 \in X_{i-1} \setminus X_{i+1}$, where $\{w_1, w_2\} \subseteq X_0 \cup \{w^*\}$. From $w_1 \in X_{i-3}$ and $w_2 \in X_{i-1}$, it follows $w_1, w_2 \not\in X_{i-2}$. By Claim 1 (3), $w_1, w_2 \not\in X_{i+1} \cup X_{i+3}$. Hence,

$$X_{i+2} \cup X_{i+1} = X_i \cup X_{i+1} = (X_0 \cup X_1 \cup \{w^*\}) \setminus \{w_1, w_2\}.$$ 

If $v_{i+3} \in A_2$, by Observation A (3), we have $X_{i+1} \subseteq X_{i+3}$, implying $w_1$ or $w_2$ is in $X_{i+3} \setminus X_{i+1}$, a contradiction. Hence, $v_{i+3} \in A_1$. Again by Claim 1 (2),
w₁ or w₂ must be in Xᵢ₊₃ \ Xᵢ₊₁, a contradiction.

By Claims 1 and 2, we have 2q + 1 = |A₁| + |A₂| ≥ 3(k + 2) = 3k + 6, contradicting q ≤ \( \frac{3k+3}{2} \). This completes the proof for q ≤ \( \frac{3k+3}{2} \).

Now consider q ≥ \( \frac{3k+5}{2} \). Observe that if q′ ≤ q, then µ(C₂q₊₁) admits a homomorphism to µ(C₂q′₊₁), which implies that \( \chi_k(µ(C₂q₊₁)) \leq \chi_k(µ(C₂q′₊₁)) \). Thus to prove the case q ≥ \( \frac{3k+5}{2} \), it suffices to give a k-fold colouring \( f \) for µ(C₃k₊6) using colours from the set \([0, 2k + \frac{k-1}{2}]\). We give such a colouring \( f \) below by using the above proof. For instance, combining Claims 1 and 2, we have 2 \( \{a, a + 1, a + 2, a + 4, \ldots, b - 2, b\} \) (mod 3k + 6). For 0 ≤ j ≤ 2k + 1, define:

\[
V[j] = \begin{cases}
< 5 + 6j, 2 + 6j >, & j = 0, 1, \ldots, \frac{k-3}{2}; \\
< 8 + 6(j - \frac{k-1}{2}), 5 + 6(j - \frac{k-1}{2}) >, & j = \frac{k-1}{2}, \ldots, k - 2; \\
< 2, 3k - 1 > \cup \{3k + 2, 3k + 5\}, & j = k - 1; \\
< 7 + 6(j - k), 6(j - k) >, & j = k, k + 1, \ldots, k + \frac{k-1}{2}; \\
< 10 + 6(j - k - \frac{k+1}{2}), 3 + 6(j - k - \frac{k+1}{2}) >, & j = k + \frac{k+1}{2}, \ldots, 2k; \\
< 4, 3k + 3 >, & j = 2k + 1.
\end{cases}
\]

Define \( f \) on C₃k₊6 by j ∈ f(vᵢ) whenever i ∈ V[j]. Observe, for each i, |f(vᵢ₋₁) ∪ f(vᵢ₊₁)) ∩ [k, 2k + 1]| ≤ \( \frac{k+1}{2} \).

Finally, let f(\( \overline{v_i} \)) be any k colours from \([k, 2k + \frac{k-1}{2}]\) \( \setminus \) (f(vᵢ₋₁) ∪ f(vᵢ₊₁)). It is straightforward to verify that \( f \) is a k-fold \( (2k + \frac{k+1}{2}) \)-colouring for µ(C₃k₊6). We shall leave the details to the reader. This completes the proof of Theorem 12. ■
References


