Distance graphs with missing multiples in the distance sets

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Abstract

Given positive integers $m, k$ and $s$ with $m > ks$, let $D_{m,k,s}$ represent the set \{1, 2, \cdots, m\} \setminus \{k, 2k, \cdots, sk\}. The distance graph $G(Z, D_{m,k,s})$ has as vertex set all integers $Z$ and edges connecting $i$ and $j$ whenever $|i - j| \in D_{m,k,s}$. The chromatic number and the fractional chromatic number of $G(Z, D_{m,k,s})$ are denoted by $\chi(Z, D_{m,k,s})$ and $\chi_f(Z, D_{m,k,s})$, respectively. For $s = 1$, $\chi(Z, D_{m,k,1})$ was studied by Eggleton, Erdős and Skilton [6], Kenmiz and Kolberg [12], and Liu [13], and was solved lately by Chang, Liu and Zhu [2] who also determined $\chi_f(Z, D_{m,k,1})$ for any $m$ and $k$. This article extends the study of $\chi(Z, D_{m,k,s})$ and $\chi_f(Z, D_{m,k,s})$ to general values of $s$. We prove $\chi_f(Z, D_{m,k,s}) = \chi(Z, D_{m,k,s}) = k$ if $m < (s + 1)k$; and $\chi_f(Z, D_{m,k,s}) = (m + sk + 1)/(s + 1)$ otherwise. The latter result provides a good lower bound for $\chi(Z, D_{m,k,s})$. A general upper bound for $\chi(Z, D_{m,k,s})$ is found. We prove the upper bound can be improved to $[(m + sk + 1)/(s + 1)] + 1$ for some values of $m, k$ and $s$. In particular, when $s + 1$ is prime, $\chi(Z, D_{m,k,s})$ is either $[(m + sk + 1)/(s + 1)]$ or $[(m + sk + 1)/(s + 1)] + 1$. By using a special coloring method called the pre-coloring method, many distance graphs $G(Z, D_{m,k,s})$ are classified into

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these two possible values of $\chi(Z, D_{m,k,s})$. Moreover, complete solutions of $\chi(Z, D_{m,k,s})$ for several families are determined including the case $s = 1$ (solved in [2]), the case $s = 2$, the case $(k, s + 1) = 1$, and the case that $k$ is a power of a prime.

Keywords. Distance graph, chromatic number, fractional chromatic number, pre-coloring method.

1 Introduction

Given a set $D$ of positive integers, the distance graph $G(Z, D)$ has all integers as vertices; and two vertices are adjacent if and only if their difference falls within $D$, that is, the vertex set is $\mathbb{Z}$ and the edge set is $\{uv : |u - v| \in D\}$. We call $D$ the distance set. The chromatic number of $G(Z, D)$ is denoted by $\chi(Z, D)$.

For different types of distance sets $D$, the problem of determining $\chi(Z, D)$ has been studied extensively. (See [2, 3, 4, 6, 7, 8, 9, 12, 15, 17].) For instance, suppose $D$ is a subset of prime numbers and $\{2, 3\} \subseteq D$, Eggleton, Erdős and Skilton [9] proved that $\chi(Z, D)$ is either 3 or 4. The problem of classifying $G(Z, D)$ with distance sets $D$ of primes into chromatic number 3 or 4 was studied by Eggleton, Erdős and Skilton [9], and by Voigt and Walther [16]. However, a complete classification is not obtained yet.

If $D$ has only one element, it is trivial that $\chi(Z, D) = 2$. When $D$ has two elements, it is known that $\chi(Z, D) = 3$ if the two integers in $D$ are of different parity, and $\chi(Z, D) = 2$ otherwise (assuming that $\gcd D = 1$). The case if $D$ has three elements, which is much more complicated, has been studied by Chen, Chang, and Huang [3], and by Voigt [15], and was solved lately by Zhu [17].

A fractional coloring of a graph $G$ is a mapping $h$ from $\mathcal{I}(G)$, the set of all independent sets of $G$, to the interval $[0, 1]$ such that $\sum_{I \in \mathcal{I}(G), x \in I} h(I) \geq 1$ for each vertex $x$ of $G$. The fractional chromatic number $\chi_f(G)$ of $G$ is the infimum of the value $\sum_{I \in \mathcal{I}(G)} h(I)$ of a fractional coloring $h$ of $G$. The fractional chromatic number of
a distance graph $G(Z, D)$ is denoted by $\chi_f(Z, D)$.

For any graph $G$, it is well-known and easy to verify that

$$\max\{\omega(G), \frac{|V(G)|}{|\alpha(G)|}\} \leq \chi_f(G) \leq \chi(G), \quad (*)$$

where $\omega(G)$ is the size (number of vertices) of a maximum complete graph, and $\alpha(G)$ is the size of a maximum independent set in $G$. (See Chapter 3 of [14].)

Given integers $m, k$ and $s$ with $m > ks$, let $D_{m,k,s}$ denote the distance set $D_{m,k,s} = \{1, 2, 3, \cdots, m\} - \{k, 2k, 3k, \cdots, sk\}$. This article studies the chromatic number and the fractional chromatic number of $G(Z, D_{m,k,s})$. If $s = 1$, the chromatic number of $G(Z, D_{m,k,1})$ was first studied by Eggleton, Erdős and Skilton [6] who determined $\chi(Z, D_{m,k,1})$ completely for $k = 1$, and partially for $k = 2$. The same results for the case $k = 1$ were also obtained in [12] by a different approach. For the cases that $k$ is an odd number, $k = 2$ and $k = 4$, $\chi(Z, D_{m,k,1})$ were determined in [13]. Recently, the exact values of $\chi_f(Z, D_{m,k,1})$ and $\chi(Z, D_{m,k,1})$ for all $m$ and $k$ were settled in [2]. We extend the study to general values of $s$.

Note that it becomes an easy case if $m < (s + 1)k$. Define a coloring $f$ of $G(Z, D_{m,k,s})$ by: For any $x \in Z$, $f(x) = x \mod k$. Since $D_{m,k,s}$ contains no multiples of $k$, $f$ is a proper coloring. Thus, $\chi(Z, D_{m,k,s}) \leq k$. As any consecutive $k$ vertices in $G(Z, D_{m,k,s})$ form a complete graph, by $(*)$, $\chi_f(Z, D_{m,k,s}) \geq k$. This implies $\chi(Z, D_{m,k,s}) = \chi_f(Z, D_{m,k,s}) = k$, if $m < (s + 1)k$. Therefore, throughout the article, we assume $m \geq (s + 1)k$.

Section 2 determines the fractional chromatic number of $G(Z, D_{m,k,s})$ for all values of $m, k$ and $s$ with $m \geq (s + 1)k$. This result provides a good lower bound for $\chi(Z, D_{m,k,s})$, namely,

$$[(m + sk + 1)/(s + 1)] \leq \chi(Z, D_{m,k,s}), \quad \text{if } m \geq (s + 1)k. \quad (**)$$

This lower bound will be shown to be sharp for some families of $G(Z, D_{m,k,s})$ and strict for some others.
Section 3 introduces the pre-coloring method, one of the main tools used in the article. For such a coloring method, we determine when it produces a proper coloring for $G(Z, D_{m,k,s})$, and then determine the number of colors used by the produced proper coloring. These characterizations are used intensively in Sections 4 and 5.

Section 4 starts with the result of a general upper bound of $\chi(Z, D_{m,k,s})$. For some values of $m$, $k$ and $s$, we improve the upper bound to $\lceil (m + sk + 1)/(s + 1) \rceil + 1$. Combining these results with the lower bound (**), mentioned above, the chromatic numbers for many families of $G(Z, D_{m,k,s})$ are determined.

Section 5 focuses on the study of $\chi(Z, D_{m,k,s})$ when $s + 1$ is a prime number. Using the results obtained in earlier sections, we show that when $s + 1$ is prime, $\chi(Z, D_{m,k,s})$ is either $\lceil (m + sk + 1)/(s + 1) \rceil$ or $\lceil (m + sk + 1)/(s + 1) \rceil + 1$. For many families of $G(Z, D_{m,k,s})$, we classify their chromatic numbers into one of these two values. Moreover, we completely determine the exact values of $\chi(Z, D_{m,k,s})$ for the following cases: If $s = 1$ (which was solved recently in [2]); if $s = 2$; if $(k, s + 1) = 1$; and if $k$ is a power of a prime.

2 Lower bounds and fractional chromatic number

In this section, we first determine the fractional chromatic number of $G(Z, D_{m,k,s})$ for all values of $m$, $k$ and $s$ with $m \geq (s + 1)k$. This result immediately leads to (**), a lower bound for $\chi(Z, D_{m,k,s})$. Then we prove that in (**), equality holds for some values of $m$, $k$ and $s$; while strict inequality holds for some others.

**Theorem 1** For any given integers $m$, $k$ and $s$ with $m \geq (s + 1)k$,

$$\chi_f(Z, D_{m,k,s}) = (m + sk + 1)/(s + 1).$$

**Proof.** For any $i$ with $0 \leq i \leq m + sk$, let $I_i = \{j \in Z : j - i \equiv xk \pmod{m + sk + 1}, 0 \leq x \leq s\}$. It is straightforward to verify that $I_i$ is an independent set in
$G(Z, D_{m,k,s})$. It is also easy to verify that any integer is contained in exactly $s + 1$

such independent sets. Define a mapping $h : \mathcal{I}(G(Z, D_{m,k,s})) \to [0, 1]$ by

$$h(I) = \begin{cases} 
\frac{1}{s+1}, & \text{if } I = I_i \text{ for } 0 \leq i \leq m + sk; \\
0, & \text{otherwise.}
\end{cases}$$

Then $h$ is a fractional coloring of $G(Z, D_{m,k,s})$ which has value $\frac{m + sk + 1}{s+1}$. Thus, $\chi_f(Z, D_{m,k,s}) \leq \frac{m + sk + 1}{s+1}$.

To show $\chi_f(Z, D_{m,k,s}) \geq \frac{m + sk + 1}{s+1}$, let $G$ be the subgraph of $G(Z, D_{m,k,s})$ induced by the vertices $\{0, 1, 2, \cdots, m + sk\}$. Then $\chi_f(G) \leq \chi_f(Z, D_{m,k,s})$. It is straightforward to verify that $\alpha(G) = s + 1$. Hence, by (\text{*}), $\chi_f(G) \geq \frac{|V(G)|}{\alpha(G)} = \frac{m + sk + 1}{s+1}$. This completes the proof of Theorem 1. Q.E.D.

Since $\chi(G)$ is an integer, by (\text{*}), we have $\lceil \chi_f(G) \rceil \leq \chi(G)$. Hence, the following is obtained.

**Theorem 2** For any given integers $m, k$ and $s$ with $m \geq (s+1)k$,

$$\chi(Z, D_{m,k,s}) \geq \lceil (m + sk + 1)/(s + 1) \rceil.$$

The following result indicates that the lower bound of $\chi(Z, D_{m,k,s})$ in Theorem 2 is attained by some values of $m, k$ and $s$, but not attained by some others.

**Theorem 3** Suppose $m \geq (s+1)k$, $k = (s+1)^a k'$ and $m + sk + 1 = (s+1)^b m'$, where both $k'$ and $m'$ are not divisible by $s + 1$. Then

$$\chi(Z, D_{m,k,s}) \begin{cases} 
\geq \frac{(m + sk + 1)}{(s + 1)} + 1, & \text{if } 0 < b \leq a; \\
= \frac{(m + sk + 1)}{(s + 1)}, & \text{if } a < b \text{ and } (s + 1, k') = 1.
\end{cases}$$

**Proof.** Let $n = (m + sk + 1)/(s + 1)$. Because $b > 0$, $n$ is an integer.

Suppose $0 < b \leq a$, we shall show that $G(Z, D_{m,k,s})$ is not $n$-colorable. Assume to the contrary, there exits an $n$-coloring $f$ of $G(Z, D_{m,k,s})$.

For any two integers $i$ and $j$, let $G[i, j]$ be the subgraph of $G(Z, D_{m,k,s})$ induced by the vertex set $\{i+1, i+2, \cdots, j\}$. Then for any integer $i$, the graph $G[i, i+m+sk+1]$
has \( m + sk + 1 \) vertices and a maximum independent set of size \( s + 1 \). Since \( f \) is an \((m + sk + 1)/(s + 1)\)-coloring, exactly \( s + 1 \) vertices of \( G[i, i + m + sk + 1] \) are colored by the same color. It follows that \( f(i) = f(i + m + sk + 1) \) for any integer \( i \).

Define a circulant graph \( G \) on the set \( \{0, 1, \ldots, m + sk\} \) with generating set \( D_{m,k,s} \), that is, \( ij \) is an edge of \( G \) if and only if \((j - i) \mod (m + sk + 1) \in D_{m,k,s} \) or \((i - j) \mod (m + sk + 1) \in D_{m,k,s} \). The argument in the previous paragraph shows that \( f \) induces a proper \( n \)-coloring of \( G \). Moreover, each color class consists of \( s + 1 \) vertices in \( G \). It is not difficult to verify that all \((s + 1)\)-independent sets of \( G \) are of the form \( \{i, i+k, \ldots, i+sk\} \). (Here each number is calculated by modulo \( m + sk + 1 \).)

Let \( d = (k, m + sk + 1) \) and \( u = (m + sk + 1)/d \). Divide the vertex set of \( G \) into \( d \) subsets of the form \( \{i, i+k, i+2k, \ldots, i+(u-1)k\} \mod (m + sk + 1) \), each of size \( u \). Then each of these \( d \) subsets is the union of some color classes of size \( s + 1 \), so \((s + 1)\) divides \( u \). Therefore \( m + sk + 1 \) is a multiple of \((s + 1)^{a+1}\), which is impossible since \( b \leq a \).

Suppose \( a < b \) and \( (s + 1, k') = 1 \), then \( u \) is a multiple of \( s + 1 \). One can easily define a proper \( n \)-coloring \( f \) on \( G \) by using \( u/(s + 1) \) colors to each of the subsets \( \{i, i+k, i+2k, \ldots, i+(u-1)k\} \mod (m + sk + 1) \) as defined in the previous paragraph by: the first \( s + 1 \) vertices in a subset use one color and the next \( s + 1 \) vertices use the next, and continue the process until all vertices are colored. It is easy to check that \( f \) is a proper coloring of \( G \). Furthermore, \( f \) can be extended to a proper coloring of \( G(Z, D_{m,k,s}) \) by letting \( f'(y) = f(x) \), where \( x = y \mod (m + sk + 1) \). Therefore, \( G(Z, D_{m,k,s}) \) is \( n \)-colorable. This completes the proof of Theorem 3. Q.E.D.

### 3 The pre-coloring method

This section introduces the main tool to be used in the remaining part of this article, namely, the **pre-coloring method**. A simpler version of this method was originally applied in [2] in determining the chromatic number of \( G(Z, D_{m,k,1}) \). Here we extend
the idea to a more complex version and use it extensively throughout this article.

Before introducing the pre-coloring method, we note another fact. Let \( Z^* \) denote the set of non-negative integers. It is known and easy to verify that for any distance set \( D \), \( \chi(Z, D) = \chi(Z^*, D) \), where \( G(Z^*, D) \) is the subgraph of \( G(Z, D) \) induced by \( Z^* \). Therefore, to color the graph \( G(Z, D_{m,k,s}) \), it suffices to color the subgraph of \( G(Z, D_{m,k,s}) \) induced by \( Z^* \).

There are two steps in the pre-coloring method. First, we partition the set \( Z^* \) into \( s + 1 \) parts by a mapping \( c : Z^* \to \{0, 1, 2, \cdots, s\} \). Second, for each non-negative integer \( x \), according to the value of \( c(x) \), we assign a color to \( x \) by the rule defined as follows.

**Definition 4** Suppose \( m, k, s \) are positive integers. For a given mapping \( c : Z^* \to \{0, 1, 2, \cdots, s\} \), define a coloring \( c' \) of \( Z^* \) recursively by:

\[
c'(j) = \begin{cases} 
  j, & \text{if } j < k; \\
  c'(j - k), & \text{if } j \geq k \text{ and } c(j) \neq 0; \\
  n, & \text{if } j \geq k \text{ and } c(j) = 0,
\end{cases}
\]

where \( n \) is the smallest non-negative integer (color) not been used in the \( m \) vertices preceding \( j \), that is, \( n = \min\{t \in Z^* : c'(j - i) \neq t \text{ for } i = 1, 2, \cdots, m\} \).

Note that \( c' \) defined above is uniquely determined by \( c \). We call \( c \) the pre-coloring, and \( c' \) the coloring induced by \( c \). For any \( x \in Z^* \), \( c(x) \) and \( c'(x) \) are called the pre-color and the color of \( x \), respectively.

In order to ensure that the coloring \( c' \) in Definition 4 to be a proper coloring for \( G(Z^*, D_{m,k,s}) \) as desired, the pre-coloring \( c \) needs to satisfy certain conditions specified in the following lemma.

**Lemma 5** Suppose \( c \) is a pre-coloring of \( Z^* \). If for any integer \( j \geq sk \), \( c(j), c(j - k), c(j - 2k), \cdots, c(j - sk) \) are all distinct, then the induced coloring \( c' \) is a proper coloring for \( G(Z, D_{m,k,s}) \).
Proof. It is enough to show by induction that for any \( j \in \mathbb{Z} \), \( c'(j) \neq c'(x) \) for any neighbor \( x \) of \( j \) and \( x < j \). If \( j < k \), or \( j \geq k \) with \( c(j) = 0 \), then this is true by Definition 4.

Now, assume \( j \geq k \) and \( c(j) \neq 0 \). By definition, \( c'(j) = c'(j - k) \). If \( j - k < x < j \), then \( x \) is adjacent to \( j - k \). By the inductive hypotheses, \( c'(x) \neq c'(j - k) \), so \( c'(x) \neq c'(j) \). If \( x < j - k \) and \( x \) is adjacent to \( j \), then either \( x \) is a neighbor of \( j - k \) or \( x = j - (s + 1)k \). In the former case, according to the inductive hypotheses, \( c'(x) \neq c'(j - k) \), hence \( c'(x) \neq c'(j) \). We now consider the case that \( x = j - (s + 1)k \). Because the pre-colors of \( j, j - k, j - 2k, \ldots, j - sk \) are all distinct, exactly one of them is 0. Suppose \( c(j - uk) = 0 \) for some \( 0 \leq u \leq s \). Then by Definition 4, \( c'(j - uk) \) is different from the color of any of the \( m \) vertices preceding \( j - uk \), hence \( c'(j - uk) \neq c'(j - (s + 1)k) \). Because \( c(j), c(j - k), \ldots, c(j - (u - 1)k) \neq 0 \), \( c'(j) = c'(j - k) = c'(j - 2k) = \cdots = c'(j - uk) \). Therefore, \( c'(j) \neq c'(j - (s + 1)k) \), i.e., \( c'(j) \neq c'(x) \). This completes the proof of Lemma 5. Q.E.D.

After getting a necessary condition for the pre-coloring \( c \) to produce a proper coloring \( c' \) for the distance graph \( G(\mathbb{Z}^*, D_{m,k,s}) \), the next natural question to ask is how many colors are used by \( c' \). The answer of this question is shown in the following result.

Lemma 6 Suppose \( c \) is a pre-coloring and \( c' \) is the induced coloring. Then the number of colors used by \( c' \) is at most \( k + \ell \), where \( \ell \) is the maximum number of vertices with pre-color 0, among any \( m - k + 1 \) consecutive integers greater than \( k \).

Proof. We prove, by induction on \( j \), that vertices 0, 1, 2, \ldots, \( j \) are colored by the pre-coloring method with at most \( k + \ell \) colors. This is trivial when \( j < k \), or \( j \geq k \) with \( c(j) \neq 0 \).

Now we assume \( j > k \) and \( c(j) = 0 \). It suffices to show that the \( m \) vertices preceeding \( j \) use at most \( k + \ell - 1 \) colors. For the \( m \) vertices preceeding \( j \), the first \( k \)
vertices use at most \( k \) colors. Among the remaining \( m-k \) vertices, only those vertices with pre-color 0 require a new color. Due to the facts that \( c(j) = 0 \), and any set of consecutive \( m-k+1 \) vertices contains at most \( \ell \) vertices of pre-color 0, we conclude that among the remaining \( m-k \) vertices, there are at most \( \ell - 1 \) vertices with pre-color 0. Therefore, the total number of colors used by the \( m \) vertices preceeding \( j \) is at most \( k + \ell - 1 \), and hence there is a color for the vertex \( j \). Q.E.D.

Combining Lemmas 5 and 6, we arrive at the following useful conclusion.

**Corollary 7** Given integers \( m, k \) and \( s \), \( \chi(Z, D_{m,k,s}) \leq n \) if there exists a pre-coloring \( c \) such that the following two conditions are satisfied:

1. for any integer \( j \geq sk \), \( c(j), c(j-k), c(j-2k), \ldots, c(j-sk) \) are all distinct, and
2. among any consecutive non-negative \( m-k+1 \) integers, there are at most \( n-k \) vertices with pre-color 0.

Corollary 7 will be used in many of the proofs in the rest of the article. Instead of finding a proper coloring for the distance graph \( G(Z, D_{m,k,s}) \) with \( n \) colors, it is enough to present a pre-coloring \( c \) that satisfies (1) and (2) of Corollary 7.

### 4 Upper bounds

This section shows upper bounds of \( \chi(Z, D_{m,k,s}) \) for different values of \( m, k \) and \( s \). Combining these upper bounds with the lower bounds obtained in Section 2 gives the exact value of \( \chi(Z, D_{m,k,s}) \) for some families of \( G(Z, D_{m,k,s}) \). In particular, we prove for many different combinations of \( m, k \) and \( s \), \( \chi(Z, D_{m,k,s}) \) is either \( [(m+sk+1)/(s+1)] \) or \( [(m+sk+1)/(s+1)] + 1 \).

We start with a general upper bound in the following. For any two integers \( a \) and \( b \), let \( (a,b) \) denote the greatest common divisor of \( a \) and \( b \).
Theorem 8 Suppose \( m \geq (s + 1)k \) and \((k, m + sk + 1) = d\), then \( \chi(Z, D_{m,k,s}) \leq d[(m + sk + 1)/d(s + 1)] \).

Proof. Define a circulant graph \( G \) on the set \( \{0, 1, \ldots, m + sk\} \) with generating set \( D_{m,k,s} \), that is, \( ij \) is an edge of \( G \) if and only if \((j - i) \mod (m + sk + 1) \in D_{m,k,s}\). It is easy to verify that any proper coloring \( f \) of \( G \) can be extended to a proper coloring \( f' \) of \( G(Z, D_{m,k,s}) \) by letting \( f'(y) = f(x) \), where \( x = y \mod (m + sk + 1) \). Therefore, it is enough to find a proper \( n \)-coloring of \( G \), where \( n = d[(m + sk + 1)/d(s + 1)] \).

Let \( u = (m + sk + 1)/d \). Divide the vertex set of \( G \) into \( d \) subsets such that each subset has \( u \) vertices and is of the form \( \{i, i + k, i + 2k, \ldots, i + (u - 1)k\} \) \( \mod (m + sk + 1) \). Any consecutive \( s + 1 \) vertices in a subset are independent, so each subset can be partitioned into \( \lfloor u/(s + 1) \rfloor = \lfloor (m + sk + 1)/d(s + 1) \rfloor \) independent sets of size \( s + 1 \), except the last one whose size might be smaller than \( s + 1 \). Therefore the vertex set of \( G \) can be partitioned into \( d[(m + sk + 1)/d(s + 1)] \) independent sets. Hence \( \chi(Z, D_{m,k,s}) \leq d[(m + sk + 1)/d(s + 1)] \). Q.E.D.

Combining the upper bound above with the lower bound in Theorem 2, the following two results emerge.

Corollary 9 Suppose \( m \geq (s + 1)k \) and \((k, m + sk + 1) = d\), then

\[
\lfloor (m + sk + 1)/(s + 1) \rfloor \leq \chi(Z, D_{m,k,s}) \leq d[(m + sk + 1)/d(s + 1)].
\]

Corollary 10 If \( m \geq (s + 1)k \) and \((k, m + sk + 1) = 1\), then \( \chi(Z, D_{m,k,s}) = \lfloor (m + sk + 1)/(s + 1) \rfloor \).

We note that in Corollary 9, there may exist big gaps between the upper and the lower bounds, depending on the values of \( d = (k, m + sk + 1) \). However, so far we do not have any example of distance graph \( G(Z, D_{m,k,s}) \) with chromatic number
exceeding \( [(m + sk + 1)/(s + 1)] + 1 \). The next theorem provides a better upper bound for some families of \( G(Z, D_{m,k,s}) \).

**Theorem 11** If \( m \geq (s + 1)k \) and \( s + 1 \) is a divisor of \( k \), then \( \chi(Z, D_{m,k,s}) \leq [(m + sk + 1)/(s + 1)] + 1 \).

**Proof.** For any \( j \in Z^* \), we can write \( j \) uniquely in the form \( j = uk + v(s + 1) + w \), where \( u, v \) and \( w \) are integers such that \( 0 \leq v < k/(s + 1) \) and \( 0 \leq w \leq s \). Then define a pre-coloring \( c \) by \( c(j) = u + w \mod (s + 1) \). We only need to show that \( c \) satisfies (1) and (2) in Corollary 7, with \( n = [(m + sk + 1)/(s + 1)] + 1 \).

First we show that for any vertex \( j \), the \( s + 1 \) vertices, \( j, j - k, j - 2k, \ldots, j - sk \) have distinct pre-colors. Assume \( j = uk + v(s + 1) + w \) with \( 0 \leq v < k/(s + 1) \) and \( 0 \leq w \leq s \). Then \( j - ik = (u - i)k + v(s + 1) + w \), \( 0 \leq i \leq s \). It follows that \( c(j - ik) = (u - i + w) \mod (s + 1) \) which give distinct colors for \( 0 \leq i \leq s \).

Next we show that among any consecutive \( m - k + 1 \) vertices, there are at most \( n - k = [(m - k + 1)/(s + 1)] + 1 \) vertices with pre-color 0. Divide the set of non-negative integers into segments of length \( s + 1 \) by \( A_0 = \{0, 1, \ldots, s\}, A_1 = \{s + 1, s + 2, \ldots, 2s + 1\}, \ldots, A_i = \{i(s + 1), i(s + 1) + 1, \ldots, (i + 1)(s + 1) - 1\}, \ldots \). Then each segment \( A_i \) contains exactly one vertex of each pre-color. Indeed, it is straightforward to verify that the pre-colors of \( A_i \) are \( \{j, j + 1, \ldots, s, 0, 1, \ldots, j - 1\} \), where \( i = uk/(s + 1) + v \), \( 0 \leq v < k/(s + 1) \) and \( j = u \mod (s + 1) \). Any set of consecutive \( m - k + 1 \) vertices intersects at most \( [(m - k + 1)/(s + 1)] + 1 \) segments, so it contains at most \( [(m - k + 1)/(s + 1)] + 1 \) vertices of pre-color 0. This completes the proof. Q.E.D.

The following corollary follows from Theorems 3 and 11.

**Corollary 12** Suppose \( m \geq (s + 1)k \), \( k = (s + 1)^a k' \) and \( m + sk + 1 = (s + 1)^b m' \), where both \( k' \) and \( m' \) are not divisible by \( s + 1 \). If \( 0 < b \leq a \), then \( \chi(Z, D_{m,k,s}) = (m + sk + 1)/(s + 1) + 1 \).
The next result shows another family of $G(Z, D_{m,k,s})$ such that the chromatic number reaches the lower bound.

**Theorem 13** If $(k, s + 1) = 1$, then $\chi(Z, D_{m,k,s}) = \lceil (m + sk + 1)/(s + 1) \rceil$ for all $m \geq (s + 1)k$.

**Proof.** Define a pre-coloring $c$ by $c(j) = j \mod (s + 1)$. We prove that $c$ satisfies (1) and (2) of Corollary 7, with $n = \lceil (m + sk + 1)/(s + 1) \rceil$.

To show that for any vertex $j$, $c(j), c(j - k), c(j - 2k), \ldots$, and $c(j - sk)$ are all distinct, we assume to the contrary that $c(j - tk) = c(j - t'k)$ for some $0 \leq t < t' \leq s$. Then $j - tk \equiv j - t'k \pmod{s + 1}$, so $(t' - t)k \equiv 0 \pmod{s + 1}$. This is impossible, because $(k, s + 1) = 1$ and $0 < t' - t \leq s$.

Next we show that among any consecutive $m - k + 1$ vertices, there are at most $\lfloor (m - k + 1)/(s + 1) \rfloor$ vertices with pre-color 0. This is trivial, because the vertices of pre-color 0 are those vertices $j$ for which $j \equiv 0 \pmod{s + 1}$, so any two vertices with pre-color 0 are exactly $s + 1$ vertices apart. This completes the proof. Q.E.D.

## 5 The case $s + 1$ is prime

This section focuses on the study of $\chi_f(Z, D_{m,k,s})$ when $s + 1$ is a prime number. If $s + 1$ is prime, then either $s + 1$ is a divisor of $k$ or $(k, s + 1) = 1$. Hence by Theorems 11 and 13, $\chi(Z, D_{m,k,s})$ is either $\lceil (m + sk + 1)/(s + 1) \rceil$ or $\lceil (m + sk + 1)/(s + 1) \rceil + 1$.

In this section, assuming $s + 1$ is prime, we classify the chromatic number for most of the families of the distance graphs $G(Z, D_{m,k,s})$ into one of those two possible values.

Similarly to Theorem 3, we let $k = (s + 1)^a k'$ and $m + sk + 1 = (s + 1)^b m'$, where $k'$ and $m'$ are not divisible by $(s + 1)$. As $s + 1$ is prime, $(s + 1, k') = 1$. Therefore, the following result can be derived immediately from Theorems 3 and 13, and Corollary 12.
Theorem 14 Suppose \( m \geq (s + 1)k \), s + 1 is prime, and \( m, k, a, b \) are defined as above. Then

\[
\chi(Z, D_{m,k,s}) = \begin{cases} 
\left\lceil (m + sk + 1)/(s + 1) \right\rceil, & \text{if } a = 0 \text{ or } a < b; \\
(m + sk + 1)/(s + 1) + 1, & \text{if } 0 < b \leq a.
\end{cases}
\]

Suppose \( k \) is a power of a prime, \( k = p^a \). If \( p \neq s + 1 \), by Theorem 14, \( \chi(Z, D_{m,k,s}) = \left\lceil (m + sk + 1)/(s + 1) \right\rceil \) for all \( m \geq (s + 1)k \). If \( p = s + 1 \), that is, \( k = (s + 1)^a \), then the chromatic number of \( G(Z, D_{m,k,s}) \) can be completely determined as follows.

Corollary 15 Suppose \( m \geq (s + 1)k \), s + 1 is prime, \( k = (s + 1)^a \), and \( m + sk + 1 = (s + 1)^b m' \), where \( m' \) is not a multiple of \( s + 1 \). Then

\[
\chi(Z, D_{m,k,s}) = \begin{cases} 
\left\lceil (m + sk + 1)/(s + 1) \right\rceil, & \text{if } b = 0 \text{ or } a < b; \\
(m + sk + 1)/(s + 1) + 1, & \text{if } 0 < b \leq a.
\end{cases}
\]

Proof. By Theorem 14, we only have to show the case as \( b = 0 \), which implies \( (k, m + sk + 1) = 1 \). Hence by Corollary 10, the prove is complete. Q.E.D.

Note that when \( s + 1 \) is prime, Theorem 14 determines the value of \( \chi(Z, D_{m,k,s}) \) unless \( a > 0 \) and \( b = 0 \). Thus, for the rest of this section, we shall assume that \( a > 0 \) and \( b = 0 \), that is, \( k \) is a multiple of \( s + 1 \) but \( m + sk + 1 \) is not. Our next result completely settles the case for \( a = 1 \).

Theorem 16 Suppose \( s + 1 \) is prime, let \( m, s, k, a, b \) be integers same as defined in Theorem 3. If \( a = 1 \), then \( \chi(Z, D_{m,k,s}) = \left\lceil (m + sk + 1)/(s + 1) \right\rceil \) for all \( m \geq (s + 1)k \).

Proof. Let \( r = \left\lceil (m + sk + 1)/(s + 1) \right\rceil \mod (s + 1) \). We consider two cases.

Case 1. \( r = 0 \). There exists an integer \( \tilde{m} \geq m \) such that \( (\tilde{m} + sk + 1)/(s + 1) = \left\lceil (m + sk + 1)/(s + 1) \right\rceil \). The distance graph \( G(Z, D_{m,k,s}) \) is a subgraph of \( G(Z, D_{\tilde{m},k,s}) \), so \( \chi(Z, D_{m,k,s}) \leq \chi(Z, D_{\tilde{m},k,s}) \). Let \( \tilde{m} + sk + 1 = (s + 1)^b \tilde{m}' \), where \( \tilde{m}' \) is not divisible...
by \((s + 1)\). Since \((\bar{m} + sk + 1)/(s + 1) \equiv r \equiv 0 \pmod{s + 1}\), \(\bar{b} \geq 2 > 1 = a\). Thus by Theorems 2 and 3, we have

\[
[(m + sk + 1)/(s + 1)] \leq \chi(Z, D_{m,k,s}) \leq \chi(Z, D_{\bar{m},k,s}) = (\bar{m} + sk + 1)/(s + 1).
\]

Therefore, \(\chi(Z, D_{m,k,s}) = [(m + sk + 1)/(s + 1)]\).

**Case 2.** \(1 \leq r < s\). Since \(s + 1\) is a prime, there exists an integer \(1 \leq t \leq s\) such that \(tr \equiv 1 \pmod{s + 1}\). Define a pre-coloring \(c\) of the set \(Z^*\) with \(s + 1\) colors as follows. For each integer \(j \in Z^*\), express \(j\) uniquely in the form \(j = u(s + 1) + v\), where \(0 \leq v \leq s\). Then let \(c(j) = (ut + v) \mod (s + 1)\). We shall show that \(c\) satisfies (1) and (2) in Corollary 7 with \(n = [(m + sk + 1)/(s + 1)]\).

Let \(j \in Z^*\). Assume, contrary to (1) of Corollary 7, \(c(j - hk) = c(j - h'k)\) for some \(0 \leq h < h' \leq s\). Let \(j - hk = u(s + 1) + v\) and \(j - h'k = u'(s + 1) + v'\), then \(ut + v \equiv u't + v' \pmod{s + 1}\). Because \(a = 1\), \((s + 1)\) divides \(k\), which implies \(j - hk \equiv j - h'k \pmod{s + 1}\), so \(v = v'\). Hence, \(ut - u't \equiv 0 \pmod{s + 1}\). This is impossible because \((t, s + 1) = 1\) and \(0 < u' - u \leq s\).

Now we show that among any \(m - k + 1\) consecutive integers, there are at most \([(m - k + 1)/(s + 1)]\) vertices of pre-color 0. Similarly to the proof of Theorem 13, we divide the set \(Z^*\) into segments of length \(s + 1\) by \(A_0 = \{0, 1, \ldots, s\}\), \(A_1 = \{s + 1, s + 2, \ldots, 2s + 1\}\), \ldots, \(A_i = \{i(s + 1), i(s + 1) + 1, \ldots, (i + 1)(s + 1) - 1\}\), \ldots. Then each of the segments \(A_i\) contains exactly one vertex of each pre-color. Indeed, it is straightforward to verify that the pre-colors of the segment \(A_i\) are \(\{j, j + 1, \ldots, i, i + 1, \ldots, j - 1\}\), where \(i \equiv v \pmod{s + 1}\), \(0 \leq v \leq s\), and \(j = vt \mod (s + 1)\).

Let \(Y = \{y, y + 1, \ldots, y + m - k\}\) be a set of \(m - k + 1\) consecutive non-negative integers. Suppose \(y \in A_i\) and \(y + m - k \in A_{i'}\). If \(|Y \cap A_i| + |Y \cap A_{i'}| \geq s + 1\), then \(Y\) intersects \([(m - k + 1)/(s + 1)]\) segments. Hence \(Y\) contains at most \([(m - k + 1)/(s + 1)]\) vertices of pre-color 0.

Assume \(|Y \cap A_i| + |Y \cap A_{i'}| < s + 1\), then \(i' - i = [(m - k + 1)/(s + 1)] \equiv [(m + sk + 1)/(s + 1)] \equiv r \pmod{s + 1}\). Recall that \(tr \equiv 1 \pmod{s + 1}\). Hence, if
$A_i$ is pre-colored by colors \{\(j, j + 1, \cdots, s, 0, 1, \cdots, j - 1\)\}, then $A_{i'}$ is pre-colored by colors \{\(j + 1, j + 2, \cdots, s, 0, 1, \cdots, j\)\}. Since \(|Y \cap A_i| + |Y \cap A_{i'}| < s + 1\), we conclude that pre-color 0 is used at most once in the set \((Y \cap A_i) \cup (Y \cap A_{i'})\). Therefore, at most \(\lceil (m - k + 1)/(s + 1) \rceil\) vertices of $Y$ have pre-color 0. This completes the proof of Theorem 16.

In the next result, we write \(m - k + 1\) in the form \(m - k + 1 = u(s + 1)k + vk + p(s + 1) + q\), where \(u, v, p, q\) are integers such that \(u \geq 0\), \(0 \leq v \leq s\), \(0 \leq p < k/(s + 1)\), \(0 \leq q \leq s\). It is easy to see that the integers \(u, v, p, q\) are uniquely determined by \(m - k + 1\).

**Theorem 17** Suppose \(m \geq (s+1)k\), \(k\) is a multiple of the prime \(s+1\), but \(m+sk+1\) is not. Let \(u, v, p, q\) be integers defined as above. If \(q \leq v + 1\), then \(\chi(Z, D_{m,k,s}) = \lceil (m + sk + 1)/(s + 1) \rceil\).

**Proof.** It suffices to show that $G(Z, D_{m,k,s})$ is \(\lceil (m + sk + 1)/(s + 1) \rceil\)-colorable. Define a pre-coloring as follows. First, partition the set of $Z^*$ into blocks recursively in such a way that the first \(k\) vertices are divided into \(k - 1\) blocks with \(k - 2\) single-vertex blocks followed by one block with two vertices. Then repeat the same process to the next \(k\) vertices and so on. Next, pre-color the blocks periodically with pre-colors \(\{0, 1, 2, \cdots, s\}\), that is, every vertex in the first block is pre-colored by 0 and so on. It is enough to show that the pre-coloring satisfies (1) and (2) of Corollary 7, with \(n = \lceil (m + sk + 1)/(s + 1) \rceil\).

First we prove that for any \(j \geq sk\), the \(s + 1\) vertices \(j, j - k, \cdots, j - sk\) receive distinct pre-colors. Suppose \(0 \leq t < t' \leq s\). Let the pre-colors of \(j - tk\) and \(j - t'k\) be \(x\) and \(y\), respectively. Because \(s + 1\) divides \(k\), and \(s + 1\) is prime, we have \((s + 1, k - 1) = 1\). As \((j - tk) - (j - t'k) = (t' - t)k\) and any consecutive \(k\) vertices are divided into \(k - 1\) blocks, so \(y \equiv x + (t' - t)(k - 1) \pmod{s + 1}\). Hence, we conclude that \(x \neq y\), since \(1 \leq t' - t < s + 1\) and \((s + 1, k - 1) = 1\).
Next we prove that among any $m - k + 1$ consecutive vertices, there are at most $\left\lceil (m - k + 1)/(s + 1) \right\rceil$ vertices with pre-color 0. Given a set $Y$ of $m - k + 1$ consecutive non-negative integers, we may assume that the first two vertices of $Y$ have pre-color 0. Among the first $u(s + 1)k$ vertices of $Y$, exactly $uk$ of them have pre-color 0, because any consecutive $(s + 1)k$ vertices are evenly pre-colored, i.e., there are exactly $k$ vertices of each pre-color.

The assumption that $m + sk + 1$ is not a multiple of $s + 1$ implies that $m - k + 1$ is not a multiple of $s + 1$. Because $k$ is a multiple of $s + 1$ while $m - k + 1$ is not, $p(s + 1) + q \geq 1$. If $p(s + 1) + q \geq 2$, then among the remaining $vk + p(s + 1) + q$ vertices of $Y$, there are $v + 1$ blocks of size 2. If we remove one vertex from each of these blocks of size 2, then the remaining $vk + p(s + 1) + q - v - 1$ vertices of $Y$ are almost evenly pre-colored, that is, the numbers of vertices with same pre-colors differ by at most one. Hence at most $\left\lceil (vk + p(s + 1) + q - v - 1)/(s + 1) \right\rceil$ of them have pre-color 0. On the other hand, among the removed vertices, exactly one vertex has pre-color 0. Therefore, the total number of vertices of pre-color 0 is at most $uk + 1 + \left\lceil (vk + p(s + 1) + q - v - 1)/(s + 1) \right\rceil = \left\lceil (m - k + 1)/(s + 1) \right\rceil$. Note that the last equality is due to the assumption that $q \leq v + 1$.

Finally, we assume $p(s + 1) + q = 1$. Then it is straightforward to verify that either $v = 0$, or the pre-color of the last vertex is not 0. Consider the remaining $vk + p(s + 1) + q = vk + 1$ vertices of $Y$. If $v = 0$, then there is one vertex of pre-color 0. If the pre-color of the last vertex is not 0, then among the remaining $vk + 1$ vertices of $Y$, there are $v$ blocks of size 2. If we remove one vertex from each of these blocks of size 2, then the remaining $vk - v$ vertices of $Y$ are almost evenly pre-colored, so at most $\left\lceil (vk - v)/(s + 1) \right\rceil$ of them have pre-color 0. On the other hand, among the vertices taken away, only one has pre-color 0. Hence, there are at most $1 + \left\lceil (vk - v)/(s + 1) \right\rceil = \left\lceil (vk + 1)/(s + 1) \right\rceil$ (because $v \leq s$) vertices of pre-color 0 in the remaining $vk + 1$ vertices of $Y$. Therefore, we conclude that $Y$ has at most
\[ uk + [(vk + 1)/(s + 1)] = [(m - k + 1)/(s + 1)] \] vertices with pre-color 0. This completes the proof. Q.E.D.

**Corollary 18** Suppose \( m \geq (s+1)k \), \( k \) is a multiple of the prime \( s+1 \), but \( m+sk+1 \) is not. Let \( u,v,p,q \) be the same as defined in Theorem 17. If \( v \geq s-1 \), or \( q \leq 1 \), then \( \chi(Z, D_{m,k,s}) = [(m+sk+1)/(s+1)] \).

Note that when \( s = 1 \), then \( v \geq s-1 \) is always true, hence we have the following corollary which was proved in [2]:

**Corollary 19** Suppose \( s = 1 \), \( m \geq 2k \), \( k = 2^a k' \) and \( m + k + 1 = 2^b m' \), where \( k' \) and \( m' \) are odd. Then

\[
\chi(Z, D_{m,k,s}) = \begin{cases} 
[(m+k+1)/2], & \text{if } b = 0 \text{ or } a < b; \\
((m+k+1)/2) + 1, & \text{if } 0 < b \leq a.
\end{cases}
\]

**Proof.** The case as \( b = 0 \) follows from Corollary 18; and the case as \( b > 0 \) follows from Theorem 14. Q.E.D.

Recall that \( k = (s+1)^a k' \) where \( a \geq 1 \) and \( k' \) is not divisible by \( s+1 \), and \( m - k + 1 \) is not divisible by \( s+1 \). In order to introduce the next result, we need the following definitions and notations. For any factor \( x \) of \( k' \), define:

\[
q(x) := [(m - k + 1)/((s + 1)^a x)] \mod (s + 1); \\
m(t, x) := \max\{t(q(x) - 1) \mod (s + 1), tq(x) \mod (s + 1)\}, 1 \leq t \leq s; \\
f(x) := \min\{m(t, x) : 1 \leq t \leq s\}.
\]

Finally, define \( f := \min\{f(x) : x \text{ is a factor of } k'\} \).

Note that for given \( m, k \) and \( s \), the integer \( f \) in the above is uniquely determined. Similarly as in Theorem 17, we let \( q = (m - k + 1) \mod (s + 1) \).

**Theorem 20** Given \( m, k \) and \( s \) where \( m \geq (s+1)k \) and \( s + 1 \) is a prime, let \( f, q \) be defined as above. If \( f + q \leq s + 1 \), then \( \chi(Z, D_{m,k,s}) = [\chi_f(Z, D_{m,k,s})] = [(m + sk + 1)/(s + 1)] \).
\textbf{Proof.} Suppose }f = f(x) = m(t, x)\text{ for some factor }x\text{ of }k'\text{ and some }1 \leq t \leq s. Express any integer }j \in \mathbb{Z}^*\text{ in the following form:}

\[ j = u(s + 1)^a x + v(s + 1) + w, \]

where }u \geq 0, 0 \leq v < (s + 1)^{a-1}x\text{ and }0 \leq w \leq s.\text{

It is easy to see that for each }j\text{, the integers }u, v, w\text{ in the form above are uniquely determined by }j.\text{ Define a pre-coloring }c\text{ using the }s + 1\text{ pre-colors }\{0, 1, \cdots, s\}\text{ by }\textcolor{red}{c(j) = (ut + w) \mod (s + 1).}\text{ In order to prove }G(Z, D_{m,k,s})\text{ is }\lfloor (m + sk + 1)/(s + 1)\rfloor\text{-colorable, it suffices to show that }c\text{ satisfies (1) and (2) of Corollary 7, with }n = \lfloor (m + sk + 1)/(s + 1)\rfloor.\text{

First, let }j\text{ be any non-negative integer, we shall show that }c(j), c(j - k), c(j - 2k), \cdots, c(j - sk)\text{ are all distinct. Let }0 \leq p' < p \leq s\text{. If }j - pk = u(s + 1)^a x + v(s + 1) + w,\text{ then}

\[
\begin{align*}
j - pk &= u(s + 1)^a x + v(s + 1) + w + (p - p')k \\
&= u(s + 1)^a x + v(s + 1) + w + (p - p')(s + 1)^a k' \\
&= u'(s + 1)^a x + v(s + 1) + w.
\end{align*}
\]

Because }s + 1, k' = (p - p', s + 1) = 1,\text{ one has }u' - u, s + 1 = 1.\text{ Assume }c(j - pk) = c(j - p'k),\text{ then }ut + w \equiv u't + w \pmod{s + 1}.\text{ Hence }t(u' - u) \equiv 0 \pmod{s + 1},\text{ which is impossible, since }s + 1\text{ is prime and }t, s + 1 = (u' - u, s + 1) = 1.\text{ This proves that }c\text{ satisfies (1) of Corollary 7.}\n
Next, we prove that among any }m - k + 1\text{ consecutive integers, there are at most }\lfloor (m - k + 1)/(s + 1)\rfloor\text{ vertices with pre-color }0.\text{ Divide the vertex set }\mathbb{Z}^*\text{ evenly into segments of length }s + 1\text{ by }A_0 = \{0, 1, 2, \cdots, s\}, A_1 = \{s + 1, s + 2, \cdots, 2s + 1\}, \cdots, A_i = \{i(s + 1), i(s + 1) + 1, \cdots, (i + 1)(s + 1) - 1\}, \cdots.\text{ Then each of the segments }A_i\text{ contains exactly one vertex of each pre-color. Indeed, the pre-colors of the segment }A_i\text{ are }\{j, j + 1, \cdots, s, 0, 1, \cdots, j - 1\},\text{ where }j = ut \pmod{s + 1},\text{ and }u\text{ is the unique integer such that }i = u(s + 1)^{a-1}x + v, 0 \leq v < (s + 1)^{a-1}x.\text{

Let }Y\text{ be a set of }m - k + 1\text{ consecutive integers, }Y = \{y, y + 1, \cdots, y + m - k\}.\text{
Suppose \( y \in A_i \) and \( y + m - k \in A_{i'} \). If \( |Y \cap A_i| + |Y \cap A_{i'}| \geq s + 1 \), then \( Y \) has at most \( \lceil (m - k + 1)/(s + 1) \rceil \) vertices with pre-color 0 (cf. proof of Theorem 16).

Now we assume that \( |Y \cap A_i| + |Y \cap A_{i'}| < s + 1 \), then \( |Y \cap A_i| + |Y \cap A_{i'}| = q \). Suppose \( i = u(s+1)^{a−1}x + v \) and \( i' = u'(s+1)^{a−1}x + v' \), where \( 0 \leq v, v' < (s+1)^{a−1}x \). Then by the definition of \( q(x) \), either \( u' - u = q(x) \) or \( u' - u = q(x) - 1 \). Suppose \( \alpha = q(x)t \mod (s+1) \) and \( \beta = (q(x) - 1)t \mod (s+1) \). Then by the choice of \( x \) and \( t \), one has \( \alpha, \beta \leq f \).

Suppose the pre-colors of \( A_i \) are \( \{j, j+1, \ldots, s, 0, 1, \ldots, j-1\} \). Then the pre-colors of \( A_{i'} \) are either \( \{j + \alpha, j + \alpha + 1, \ldots, s, 0, 1, \ldots, j + \alpha - 1\} \), if \( u' - u = q(x) \); or \( \{j + \beta, j + \beta + 1, \ldots, s, 0, 1, \ldots, j + \beta - 1\} \), if \( u' - u = q(x) - 1 \).

Any other segment different from \( A_i \) and \( A_{i'} \) is either disjoint from \( Y \) or contained in \( Y \). As each segment contains exactly one vertex of each color, to prove that \( Y \) has at most \( \lceil (m - k + 1)/(s + 1) \rceil \) vertices with pre-color 0, it suffices to show that the pre-color 0 is used at most once in the union \( (Y \cap A_i) \cup (Y \cap A_{i'}) \). Assume that 0 is used in both \( Y \cap A_i \) and \( Y \cap A_{i'} \). Without loss of generality, we may assume that the pre-colors of \( A_{i'} \) are \( \{j + \alpha, j + \alpha + 1, \ldots, s, 0, 1, \ldots, j + \alpha - 1\} \). Then one has \( |Y \cap A_i| \geq j \) and \( |Y \cap A_{i'}| \geq s + 1 - (j + \alpha - 1) \). It follows that \( q = |(Y \cap A_i) \cup (Y \cap A_{i'})| \geq s + 2 - \alpha \), contrary to the assumption that \( \alpha + q \leq f + q \leq s + 1 \). Therefore \( c \) satisfies (2) of Corollary 7, with \( n = \lceil (m + sk + 1)/(s + 1) \rceil \). This completes the proof of Theorem 20. Q.E.D.

**Corollary 21** If \( m \geq (s + 1)k \), \( s + 1 \) is prime, and there is a factor \( x \) of \( k' \) such that \( q(x) \leq 1 \), then \( \chi(Z, D_{m,k,s}) = \lceil (m + sk + 1)/(s + 1) \rceil \). In particular, if \( \lceil (m - k + 1)/k \rceil \mod (s + 1) \leq 1 \), then \( \chi(Z, D_{m,k,s}) = \lceil (m + sk + 1)/(s + 1) \rceil \).

**Proof.** According to definition, if \( q(x) = 1 \), then \( m(1, x) = 1 \); if \( q(x) = 0 \), then \( m(t, x) = 1 \) for some \( t \) such that \( ts \equiv 1 \mod (s + 1) \). (Such a \( t \) exists, because
In any of the two cases, $f = 1$, so $f + q \leq s + 1$. Therefore, $\chi(Z, D_{m,k,s}) = \lceil (m + sk + 1)/(s + 1) \rceil$ by Theorem 20. Q.E.D.

Applying Theorem 14 and Corollaries 18 and 21, we are able to completely settle the case $s = 2$.

**Corollary 22** Suppose $s = 2$, $m \geq 3k$, $k = 3^a k'$ and $m + 2k + 1 = 3^b m'$, where $k'$ and $m'$ are not multiples of 3. Then

$$\chi(Z, D_{m,k,2}) = \begin{cases} 
\lceil (m + 2k + 1)/3 \rceil, & \text{if } b = 0 \text{ or } a < b; \\
(m + 2k + 1)/3 + 1, & \text{if } 0 < b \leq a.
\end{cases}$$

**Proof.** According to Theorem 14, we only have to show the case as $b = 0$. Suppose $m - k + 1 = u(s + 1)k + v + p(s + 1) + q$. If $v \neq 0$, then the conclusion follows from Corollary 18. If $v = 0$, then the conclusion follows from Corollary 21. (Because $\lfloor (m - k + 1)/k \rfloor \mod (s + 1) \leq 1$.) Q.E.D.

**Remarks.** New results related to this topic have been obtained since the submission of this paper. In [5], it was proved that $\chi(G(Z, D_{m,k,s})) \leq \lceil (m + sk + 1)/(s + 1) \rceil + 1$ for all $m \geq (s + 1)k$. Then in [11], the chromatic numbers of all the graphs $G(Z, D_{m,k,s})$ are completely determined. The circular chromatic number of the class of distance graphs $G(Z, D_{m,k,s})$ was studied in [1, 11, 19], and the value of $\chi_c(Z, D_{m,k,s})$ has been completely determined in [19]. (The circular chromatic number $\chi_c(G)$ of a graph $G$ is a refinement of $\chi(G)$, and $\chi(G) = \lceil \chi_c(G) \rceil$ for any graph $G$. For a survey of research concerning circular chromatic number of graphs, see [20].)

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**References**


